NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)
Pampady, Thiruvilwamala, Thrissur Dist, Kerala-680588

DEPARTMENT OF MECHANICAL ENGINEERING

COURSE MATERIALS

ME 202 ADVANCED MECHANICS OF SOLID

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered : B.Tech in Mechanical Engineering
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of Dr. A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Producing internationally competitive Mechanical Engineers with social responsibility & sustainable employability through viable strategies as well as competent exposure oriented quality education.

DEPARTMENT MISSION

1. Imparting high impact education by providing conductive teaching learning environment.
2. Fostering effective modes of continuous learning process with moral & ethical values.
3. Enhancing leadership qualities with social commitment, professional attitude, unity, team spirit & communication skill.
4. Introducing the present scenario in research & development through collaborative efforts blended with industry & institution.

PROGRAMME EDUCATIONAL OBJECTIVES

PEO1: Graduates shall have strong practical & technical exposures in the field of Mechanical Engineering & will contribute to the society through innovation & enterprise.

PEO2: Graduates will have the demonstrated ability to analyze, formulate & solve design engineering / thermal engineering / materials & manufacturing / design issues & real life problems.

PEO3: Graduates will be capable of pursuing Mechanical Engineering profession with good communication skills, leadership qualities, team spirit & communication skills.

PEO4: Graduates will sustain an appetite for continuous learning by pursuing higher education & research in the allied areas of technology.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation

of data, and synthesis of the information to provide valid conclusions.

5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and teamwork**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO1: graduates able to apply principles of engineering, basic sciences & analytics including multi variant calculus & higher order partial differential equations..

PSO2: Graduates able to perform modeling, analyzing, designing & simulating physical systems, components & processes.

PSO3: Graduates able to work professionally on mechanical systems, thermal systems & production systems.

COURSE OUTCOMES

CO1	Understand the methodologies in theory of elasticity at a basic level.
CO2	Differentiate constitutive relation and solve 2D problems in elasticity.
CO3	Evaluate the governing equations in cylindrical coordinate to solve Axisymmetric

	problems.
CO4	Analyze Unsymmetrical bending of beams and determine the shear centre.
CO5	Calculate the energy formulations of various elasticity problems.
CO6	Analyze torsion of circular/non circular bars using classical method.

MAPPING OF COURSE OUTCOMES WITH PROGRAM OUTCOMES

	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PSO1	PSO2	PSO3
CO1	3	2	3	3	-	-	-	-	-	-		3	3	3	0
CO2	3	2	3	2	-	-	-	-	-	-		1	3	3	0
CO3	3	3	2	2	-	-	-	-	-	-		3	3	3	0
CO4	3	3	3	2	-	-	-	-	-	-		3	3	3	0
CO5	3	3	3	2	-	-	-	-	-	-		3	3	3	0
CO6	3	3	3	2	-	-	-	-	-	-		3	3	2	0

Note: H-Highly correlated=3, M-Medium correlated=2, L-Less correlated=1

SYLLABUS

Course code	Course Name	L-T-P-Credits	Year of Introduction
ME202	ADVANCED MECHANICS OF SOLIDS	3-1-0-4	2016

Prerequisite: ME201 Mechanics of solids

Course Objectives: The main objectives of the course are

1. To impart concepts of stress and strain analyses in a solid.
2. To study the methodologies in theory of elasticity at a basic level.
3. To acquaint with the solution of advanced bending problems.
4. To get familiar with energy methods for solving structural mechanics problems.

Syllabus

Introduction, concepts of stress, equations of equilibrium, strain components, strain-displacement relations, compatibility conditions, constitutive relations, boundary conditions, 2D problems in elasticity, Airy's stress function method, unsymmetrical bending of straight beams, bending of curved beams, shear center, energy methods in elasticity, torsion of non-circular solid shafts, torsion of thin walled tubes.

Expected outcome: At the end of the course students will be able to

1. Apply concepts of stress and strain analyses in solids.
2. Use the procedures in theory of elasticity at a basic level.
3. Solve general bending problems.
4. Apply energy methods in structural mechanics problems.

Text Books:

1. L. S. Sreenath, Advanced Mechanics of Solids, McGraw Hill,2008
2. S. M. A. Kazimi, Solid Mechanics, McGraw Hill,2008
3. S. Jose, Advanced Mechanics of Materials, Pentagon Educational Services,2013
4. L. Govindaraju ,TG Sitharaman, Applied elasticity for Engineers, NPTEL
5. U. Saravanan, Advanced Solid Mechanics, NPTEL
6. S. Anil Lal, Advanced Mechanics of Solids, Siva Publications and Distributions, 2017

References Books:

1. S. P. Timoshenko, J. N. Goodier, Theory of elasticity, McGraw Hill,1970
2. R.J. Atkin, and N. Fox, An introduction the theory of elasticity, Longman,1980
3. J. P. Den Hartog, Advanced Strength of Materials, McGraw Hill,1987
4. C. K. Wang, Applied Elasticity, McGraw Hill,1983
5. www.solidmechanics.org/contents.htm - Free web book on Applied Mechanics of Solids by A.F. Bower.

Course Plan			
Module	**Contents**	**Hours**	**Sem. Exam Marks**
I	Introduction to stress analysis in elastic solids - stress at a point – stress tensor – stress components in rectangular and polar coordinate systems - Cauchy's equations – stress transformation – principal stresses and planes - hydrostatic and deviatoric stress components, octahedral shear stress - equations of equilibrium	6	15%
	Displacement field – engineering strain - strain tensor *(basics only)* – analogy between stress and strain tensors - strain-displacement relations *(small-strain only)* – compatibility conditions	4	
II	Constitutive equations – generalized Hooke's law – equations for linear elastic isotropic solids - relation among elastic constants – Boundary conditions – St. Venant's principle for end effects – uniqueness theorem	4	15%
	2-D problems in elasticity - Plane stress and plane strain problems – stress compatibility equation - Airy's stress function and equation – polynomial method of solution – solution for bending of a cantilever with an end load	4	
FIRST INTERNAL EXAM			
III	Equations in polar coordinates (2D) – equilibrium equations, strain-displacement relations, Airy's equation, stress function and stress components *(only short derivations for examination)*	3	15%
	Application of stress function to Lame's problem and stress concentration problem of a small hole in a large plate *(only stress distribution)*	3	
	Axisymmetric problems – governing equations – application to thick cylinders, rotating discs.	4	
IV	Unsymmetrical bending of straight beams *(problems having c/s with one axis of symmetry only)* – curved beams *(rectangular c/s only)* - shear center of thin walled open sections *(c/s with one axis of symmetry only)*	6	15%
	Strain energy of deformation – special cases of a body subjected to concentrated loads, moment or torque - reciprocal relation – strain energy of a bar subjected to axial force, shear force, bending moment and torque	3	
SECOND INTERNAL EXAM			
V	Maxwell reciprocal theorem – Castigliano's first and second theorems – virtual work principle – minimum potential energy theorem.	5	20%

	Torsion of non-circular bars: Saint Venant's theory - solutions for circular and elliptical cross-sections	4	
VI	Prandtl's method - solutions for circular and elliptical cross-sections - membrane analogy.	4	20%
	Torsion of thin walled tubes, thin rectangular sections, rolled sections and multiply connected sections	6	
END SEMESTER EXAM			

Question Paper Pattern

Total marks: 100, Time: 3 hrs
The question paper should consist of three parts

Part A

4 questions uniformly covering modules I and II. Each question carries 10 marks
Students will have to answer any three questions out of 4 (3 X10 marks = 30 marks)

Part B

4 questions uniformly covering modules III and IV. Each question carries 10 marks
Students will have to answer any three questions out of 4 (3 X10 marks = 30 marks)

Part C

6 questions uniformly covering modules V and VI. Each question carries 10 marks
Students will have to answer any four questions out of 6 (4 X 10 marks = 40 marks)

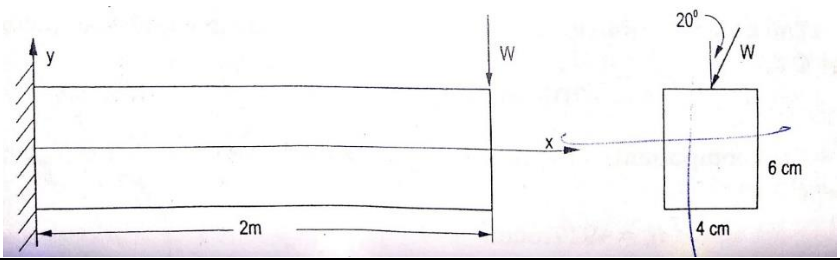
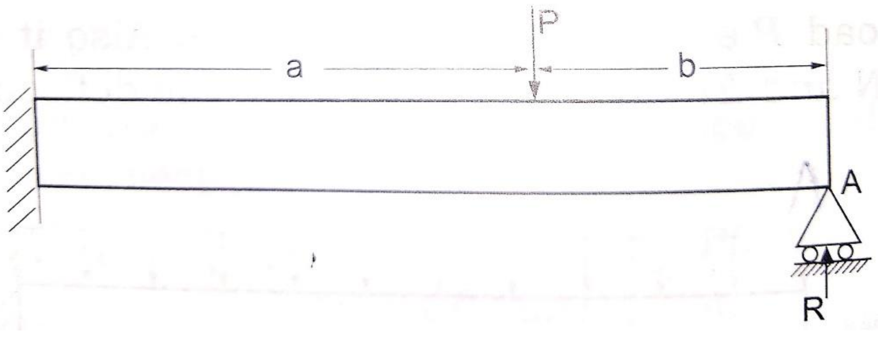
Note: In all parts, each question can have a maximum of four sub questions, if needed.

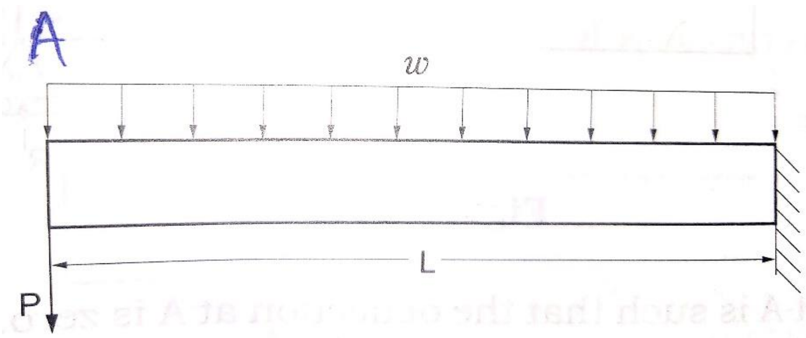
QUESTION BANK

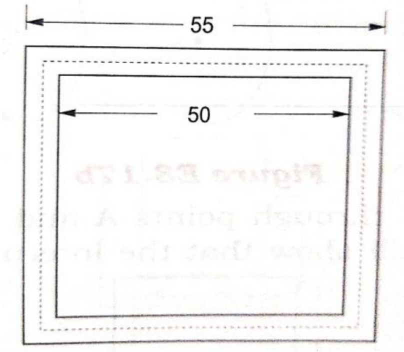
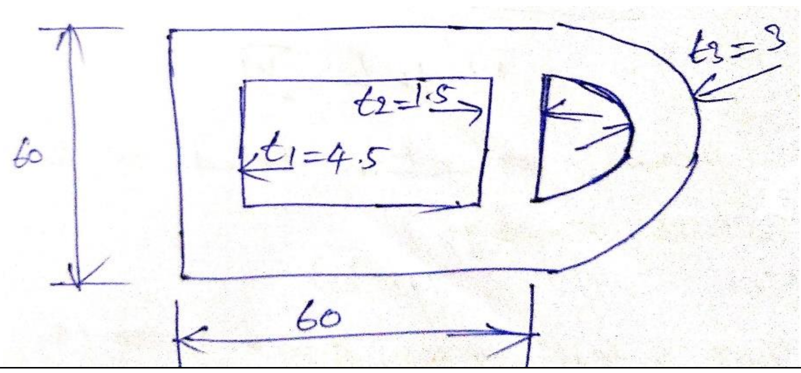
Knowledge Level (KL)	K1 : Remembering	K3:Applying	K5: Evaluating
Course Outcome (CO)	K2: Understanding	K4: Analysing	K6: Creating

MODULE I			
Q:NO:	**QUESTIONS**	**CO**	**KL**
1	Explain stress at a point in a rectangular shaped member	CO1	K2
2	What is the significance of Compatibility Condition?	CO1	K2
3	The state of stress at a point is given by the components σ_x = 70 MPa, σ_y = 10 MPa, σ_z = 20 MPa, τ_{xy} = -40 MPa, τ_{yz} = τ_{zx} = 20 MPa. Determine the value of Principal stresses , Maximum Shear stress and Maximum Principal stress directions.	CO1	K5

4	Define the followings; (a) State of stress at a point (b) Shearless plane	CO1	K2
5	Derive an equilibrium equation for plane stress state	CO1	K3
6	What are stress Invariants and Strain Invariants? Explain	CO1	K2
7	The displacement field for a body is given by U= (x^2+y) i+ (3+z) j+ (x^2+2-y)k. (a) Write down the strain tensor at the point (3,2,-1) (b). Determine the Principal strain at (3,2,-1) and the direction of maximum principal strain.	CO1	K5
8	Define the followings; (a) Hydrostatic stress (b) Deviatorial stresses	CO1	K2
	MODULE II		
1	Derive the relations between an elastic constants K,E and v	CO2	K3
2	Derive the relationship between stress and strain for an isotropic material in terms of Lame's co-efficient.	CO2	K3
3	For steel, the following data is applicable. E= 207 x 10^6 KPa and G= 80 x 10^6 KPa. For the given strain matrix at a point determine the stress matrix $$[\varepsilon_{ij}]=\begin{bmatrix} 0.01 & 0 & -0.002 \\ 0 & -0.003 & 0.0003 \\ -0.002 & 0.0003 & 0 \end{bmatrix}$$	CO2	K5
4	What are Lame's Co-efficient? How are they related to Poisson's ratio?	CO2	K2
5	State and explain generalized Hook's law	CO2	K5
6	State and explain Saint Venant's Principle for end effects with a suitable example	CO2	K3
7	State and prove Uniqueness Theorem in Theory of Elasticity.	CO2	K4
8	Define Constitutive Law	CO2	K2
	MODULE III		
1	Sketch a 2-Dimensional element in polar co-ordinate (r,θ) system and show all stresses on it.	CO3	K3
2	Draw the stress distribution around a small hole (diameter 'b'), on a thin plate having large width ('a') where b<<a, subjected to uniform tensile force at the two edges.	CO3	K2
3	Derive the equation for radial & hoop stress developed in a thick cylinder subjected to both internal and external pressure for a plane stress case.	CO3	K5
4	Sketch the circumferential stress distribution for a thick cylinder subjected to internal pressure only	CO3	K3

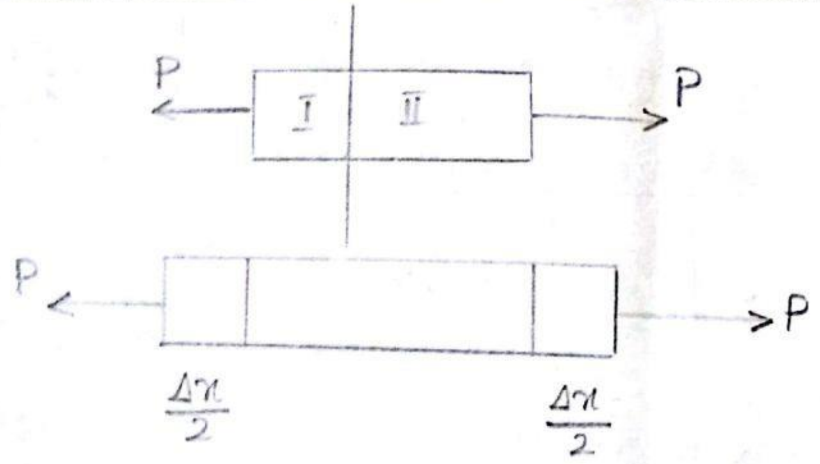
5	Describe the Airy's stress function with the help of second degree polynomial?	CO3	K3
6	Derive the equilibrium equation in 2-Dimensional Polar Co-ordinate system?	CO3	K3
7	Obtain the stress distribution in a rotating disc of radius 'b' with no external force at the outer surface	CO3	K5
8	Sketch the circumferential stress distribution for a thick cylinder subjected to internal pressure only	CO3	K3
	MODULE IV		
1	What is meant by Shear Centre?	CO4	K2
2	Explain the term "Complementary Strain Energy	CO4	K2
3	Derive the equation for strain energy in bending of cantilever beam with a point load and simply supported beam with concentrated load.	CO4	K5
4	A cantilever of rectangular cross section of breadth 4 cm and depth 6 cm is subjected to an inclined load "W" at free end. The length of cantilever is 2 m and the angle of inclination of the load with vertical is 20°. What is the maximum value of "W" if the maximum stress due to bending is not to exceed 200N/mm^2. 	CO4	K6
5	Find the support reaction "R" in figure at the end of the cantilever beam using strain energy method. (Load acting is "P" at a distance of "b" from the support). 	CO4	K6
6	Give the expression for strain energy due to torsion	CO4	K2
7	Explain Unsymmetrical Bending	CO4	K2

8	Derive the expression for stress developed in curved beam subjected to bending moment ‘M’?	CO4	K5
9	Explain the principle of virtual work in energy methods and its application in finding load and displacement at a point	CO4	K2
MODULE V			
1	State and Explain Minimum Potential Energy	CO5	K3
2	Write the general expression for twisting moment for shafts of non-circular cross section incorporating warping function $\psi(x,y)$	CO5	K4
3	The section of a square shaft is 5cm x 5 cm and a torque of 5000 Kg.Cm is applied. Determine the maximum shear stress and angle of twist per unit length.	CO5	K5
4	The cantilever beam supports a uniformly distributed load “w” and a concentrated load “P” as shown in figure. Also, it is given that L=2m, w=4 KN/m, P=6KN and EI = 5 MN.m^2.Determine the deflection at the free end (at point A) using castigliano’s theorem 	CO5	K5
5	State and prove reciprocal relation in strain energy	CO5	K3
6	Derive the equation for torsion of an elliptical cross-section using Saint Venant’s method.	CO5	K4
7	State and prove Castiliagno’s First and Second Theorem	CO5	K5
8	A rod with rectangular cross section is used to transmit torque to a machine frame(see figure). It has a width of 40 cm. The first 3.0m length of rod has a depth of 60mm and the remaining 1.5m length has a depth of 30 mm. The rod is made of steel having G= 77.5 GPa. Given T1=750 Nm and T2= 400Nm. Determine the maximum shear stress in the rod. Also, determine the angle of twist of the free end	CO5	K5
MODULE VI			
1	Explain the application of membrane analogy in solving torsion problem of prismatic bar of any cross section for finding twisting moment and shear stress acting on the cross-section.	CO6	K2

2	Find an expression for the max shear stress induced in an elliptical bar under torsion?	CO6	K4
3	Derive the torsion equation for a thin walled hollow circular rod subjected to a torque “T”. Also, state the assumptions used in the derivation	CO6	K4
4	A shaft of square section as shown in figure below is subjected to a twisting moment such that the maximum shear stress is limited to 250 GN/mm^2.Obtain the torque and angular twist, if the shaft is 1.6m long. (Take G=70000 N/mm^2) 	CO6	K2
5	Define the term Shear Flow	CO6	K2
6	What is meant by warping function	CO6	K2
7	Why closed sections are having better torsional rigidity than open sections. Briefly explain.	CO6	K4
8	Derive an expression for angle of twist per unit length for a thin walled tube subjected to a torque ‘T’	CO6	K5
9	A hollow thin wall torsion member has two compartments with cross sectional dimensions as given in figure. He material is an aluminium alloy having G=26 GPa. Determine the torque and unit angle of twist if the maximum shear stress is 40 MPa 	CO6	K5

MODULE-1

14/2/17



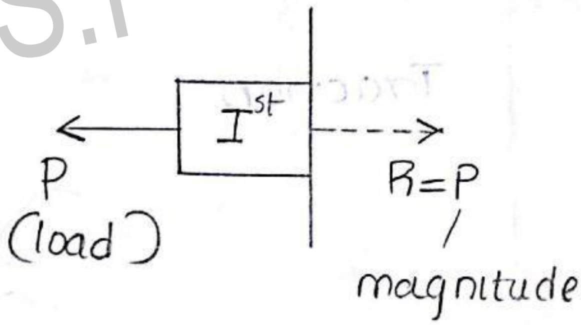
Total extension Δx

$$\text{Stress} = \text{linear stress} = \frac{\text{Resistance Force}}{\text{unit area}} = \frac{R}{A}$$

unit N/m^2

Tensile stress $= \sigma$

shearing stress $= \tau$



$$\text{strain } \varepsilon = \frac{\Delta x}{l}$$

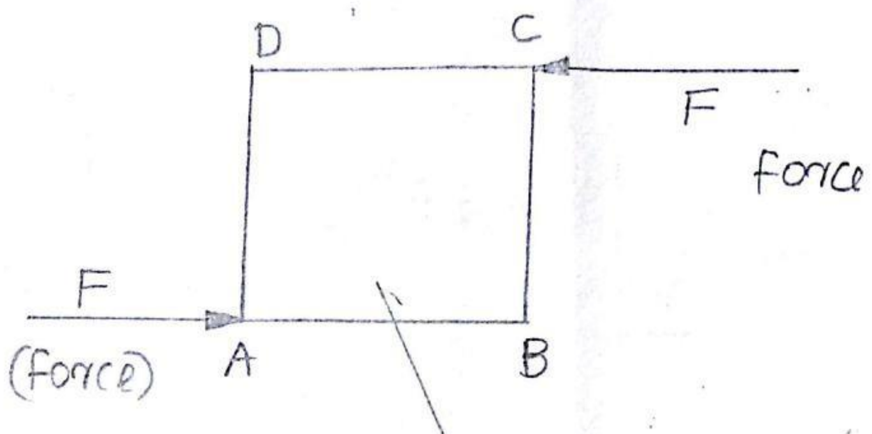
$$= \frac{\text{change in length}}{\text{original length}}$$

Tensile strain $= \varepsilon$

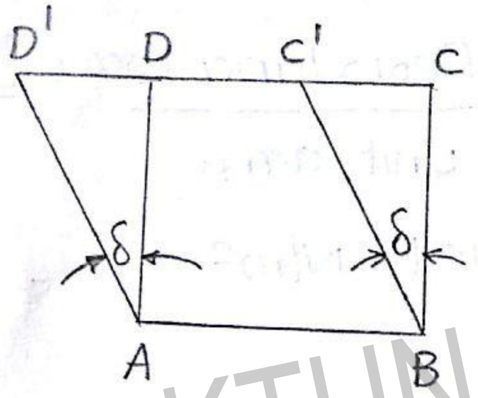
shearing strain $= \phi$

The stress $=$ (Resistance force / unit area) is also called as traction.

Shearing action



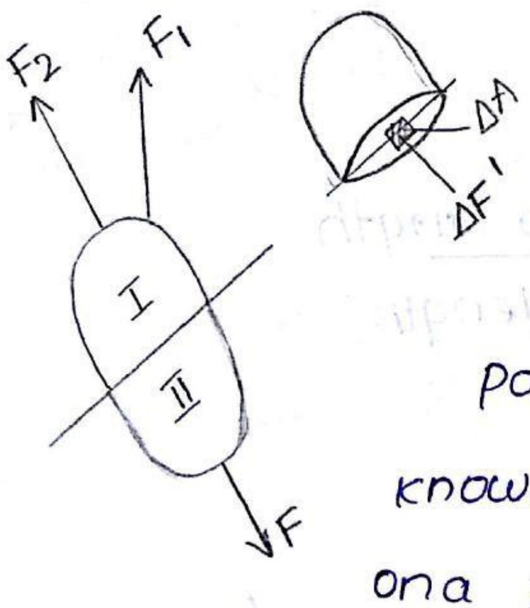
Element subjected to shear.



Shear stress $= \tau$

Shear strain $= \delta$

Traction



$$\text{Stress} = \frac{\Delta F'}{\Delta A}$$

$$\underset{\Delta A \to 0}{Lt} \frac{\Delta F'}{\Delta A} = \text{stress at a}$$

point on a plane. This is also known as "traction at a point on a plane"

$$\text{Traction at a point on a plane} = \underset{\Delta A \to 0}{Lt} \frac{\Delta F'}{\Delta A}$$

Traction at a point

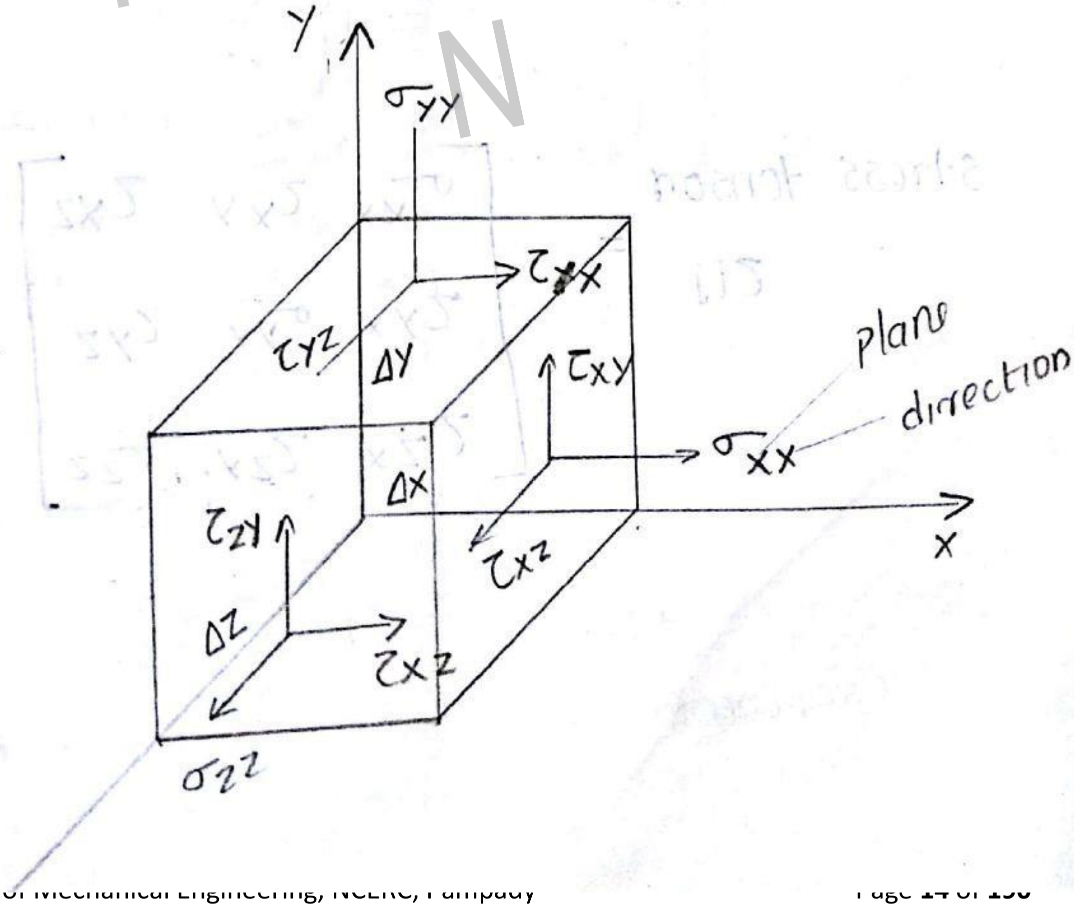
consider a solid in equelibria actd up on by one external force or system of forces. Traction at a point on a plane is the resisting force per unit area.

Traction is denoted as $\vec{T}$ and its value at a point on a plane is calculated as

$$\vec{T} = \underset{\substack{\text{small area} \to 0 \\ \text{'a'}}}{Lt} \frac{\text{Force resisted by 'a' on a plane arround a point}}{\text{small area a}}$$

$$\vec{T} = \underset{\Delta A \to 0}{Lt} \frac{\Delta F'}{\Delta A}$$

Co-ordinate plane.



Similarly we can find stress components

y and z plane.

For

x Plane σ_{xx} τ_{xy} τ_{xz}

y Plane σ_{yy} τ_{yx} τ_{yz}

z Plane σ_{zz} τ_{zx} τ_{zy}

Plane	x direction	y direction	z direction
x Plane	σ_{xx}	τ_{xy}	τ_{xz}
y Plane	τ_{yx}	σ_{yy}	τ_{yz}
z Plane	τ_{zx}	τ_{zy}	σ_{zz}

$$\text{stress tensor } \tau_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Shear component

normal stress components

Stress tensor of a point is written by combining the traction on a set of a any 3 mutually perpendicular ~~lines~~ planes passing through that point. They are also known as rectangular stress components.

$$T_P^{\hat{n}=\hat{i}} = \sigma_{xx}\hat{i} + \tau_{xy}\hat{j} + \tau_{xz}\hat{k}$$

Normal vector

$$T_P^{\hat{n}=\hat{j}} = \sigma_{yy}\hat{i} + \tau_{yx}\hat{j} + \tau_{yz}\hat{k}$$

Traction plane

Cauchy's Equations

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

Traction in x, y, z direction

Stress tensor

direction cosines of any arbitrary plane (where it is situated)

$$\vec{T} = \sqrt{T_x^2 + T_y^2 + T_z^2}$$

Resultant traction

$$\sigma_n = \vec{T} \cdot n$$

unit vector

normal stress component

$$\tau_n = \sqrt{(\vec{T})^2 - \sigma_n^2}$$

shear stress component

1\. Stress tensor $\tau_{ij} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ units

determine the traction vector and its components on an arbitrary plane having direction cosines $n_x = n_y = n_z = \frac{1}{\sqrt{3}}$

From cauchy's Equation

stress tensor × direction cosines of arbitrary plane = Components traction vector

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} + \frac{2}{\sqrt{3}} \\ \frac{3}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \frac{6}{\sqrt{3}} \\ \frac{6}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{bmatrix} \text{ units}$$

$$\vec{T} = \sqrt{T_x^2 + T_y^2 + T_z^2}$$

$$= \sqrt{\left(\frac{6}{\sqrt{3}}\right)^2 + \left(\frac{6}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2}$$

$$= \underline{5.0332}$$

Normal stress component

$$\sigma_n = \vec{T} \cdot n$$

n = unit vector

$$= (n_x \hat{i} + n_y \hat{j} + n_z \hat{k})$$

$$\sigma_n = \vec{T} \left(\frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k \right)$$

esultant

$$= \left(\frac{6}{\sqrt{3}} \hat{i} + \frac{6}{\sqrt{3}} \hat{j} + \frac{2}{\sqrt{3}} \hat{k} \right) \cdot \left(\frac{1}{\sqrt{3}} \hat{i} + \frac{1}{\sqrt{3}} \hat{j} + \frac{1}{\sqrt{3}} \hat{k} \right)$$

$$= \underline{\underline{\frac{14}{3} \text{ units.}}}$$

shear stress component

$$\tau_n = \sqrt{(\vec{T})^2 - \sigma_n^2}$$

$$= \sqrt{(5.0332)^2 - \left(\tfrac{14}{3}\right)^2}$$

$$= \underline{\underline{\cancel{24.899}\ 1.885 \text{ units}}}$$

Stress tensor is given by

$$\tau_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ Determine}$$

1. magnitude of traction for an arbitrary plane having direction cosines $n_x = \sqrt{3}/2$, $n_y = \frac{-1}{2}$ and $n_z = 0$ and also determine The components of Resultant stress.

Stress transformation

The process of converting the stress matrix of one co-ordinate system to another co-ordinate system is called stress transformation. Stress transformation can be determined by using the formula.

$$\sigma' = Q[\sigma][Q]^T$$

where

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Stress Tensor is given as $\tau_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ units when the co-ordinate is rotated by an angle $\theta = 45$. Find out corresponding new stress components. σ' ?

$$\sigma' = Q \cdot [\sigma][Q]^T$$

σ = stress tensor

$$\sigma' = \begin{bmatrix} \cos 45 & \sin 45 & 0 \\ -\sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0 & \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} + \frac{2}{\sqrt{2}} & \frac{-2}{\sqrt{2}} + \frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} & \frac{5}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} + \frac{5}{2} & \frac{-3}{2} + \frac{5}{2} & 0 \\ \frac{1}{2} + \frac{1}{2} & \frac{-1}{2} + \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{8}{2} & \frac{2}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ units}$$

The resisting traction vectors on the x, y and z plane passing through a point given below.

$\overset{\hat{i}}{T} = 3\hat{i} + 2\hat{j} - 2\hat{k}$. similarly $\overset{\hat{j}}{T} = 2\hat{i} + 0\hat{j} - \hat{k}$

(Traction on x plane) The unit of traction is kPa.

$\overset{\hat{k}}{T} = -2\hat{i} - \hat{j} + 2\hat{k}$.

1) Write down the matrix of stress tensor in cartesian co-ordinate system.

$$\text{stress tensor} = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix} \text{kPa}$$

Stress tensor is given as $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$ kPa

Evaluate the matrix of new co-ordinate system obtained by rotating the cartesian co-ordinate system through an angle 30°. in the anticlockwise direction.

10/2/17

PRINCIPLE STRESSES AND PRINCIPLE PLAIN

For every point on the solid there exist a plane on which the traction is along its normal vector. ie shear stress is equal to zero. This plane is called principle plane. Magnitude of the traction on the principle plane is called Principal stress.

$$\vec{T} = \sigma . \hat{n} \quad \text{---} (1)$$

$$[\vec{T}] = [\sigma][\hat{n}] \quad \text{---} (2)$$

$$\sigma \hat{n} \quad [\sigma][\hat{n}] = \sigma [\hat{n}].$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = 0.$$

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{vmatrix} = 0$$

$$\sigma^3 - \overbrace{[\sigma_{xx}+\sigma_{yy}+\sigma_{zz}]}^{I_1}\sigma^2 + \overbrace{\left\{ \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix} \right.}^{I_2} - \overbrace{|\tau_{ij}|}^{I_3} = 0$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

where

I_1, I_2 and I_3 are called stress invariants

. state of stress at a point is given by the cartesian stress tensor $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ KPa. Find the three stress invariants I_1, I_2 and I_3.

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$= 3+5+3 = 11 \text{ KPa}$$

$$I_2 = \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix}$$

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= (5\times3-1) + (9-1) + (15-1)$$

$$= 14 + 8 + 14$$

$$= \underline{\underline{36 \text{ KPa}}}$$

$$\tau_{ij} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

I_3

$$= \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) - 1(-3-1) + 1(1-5)$$

$$= 3 \times 14 + 1(-3+1) + (-4)$$

$$= \cancel{42 - 3 - 4} \quad 42 - 2 - 4 = 36\ \text{KPa}$$

Write down the characteristic equation.

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$\sigma^3 - 11\sigma^2 + 36\sigma - 36 = 0.$$

Determine the principal stresses

Principal stressess are the roots of the characteristics equation.

$$\sigma = 6, 3, 2$$

$$\left.\begin{aligned} \sigma_1 &= 6\ \text{KPa} \\ \sigma_2 &= 3\ \text{KPa} \\ \sigma_3 &= 2\ \text{KPa} \end{aligned}\right\}$$ Principal stressess acting on the plane.

Principal plane corresponding to ... Principal stress σ is given by

$$\begin{bmatrix} \sigma_x-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y-\sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z-\sigma \end{bmatrix}\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}=0$$

$$\begin{bmatrix} 3-6 & -1 & 1 \\ -1 & 5-6 & -1 \\ 1 & -1 & 3-6 \end{bmatrix}\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}=0$$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}=0$$

$$n_x^2+n_y^2+n_z^2=1$$

Since we already have ~~one~~ above equation, any two say 1, 2 equation along can be use to generate a equation.

$$-3n_x-n_y+n_z=0 \quad \text{---(1)}$$

$$-n_x-n_y-n_z=0 \quad \text{---(2)}$$

$$n_x-n_y-3n_z=0 \quad \text{---(3)}$$

From (1) $\quad n_z=3n_x+n_y$

(2) $\quad -n_z=n_x+n_y$

$$2n_z = 2n_x$$

$$n_z = n_x$$

substitute the values in (2)

$$-n_x - n_y - n_x = 0$$

$$-2n_x - n_y = 0$$

$$n_y = -2n_x = -2n_z$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$n_z^2 + (-2n_z)^2 + n_z^2 = 1$$

$$n_z^2 + 4n_z^2 + n_z^2 = 1$$

$$6n_z^2 = 1$$

$$n_z = \sqrt{1/6} = \frac{1}{\sqrt{6}}$$

$$n_z = \frac{1}{\sqrt{6}}$$

$$n_x = n_z = \frac{1}{\sqrt{6}}$$

$$n_y = -2n_z = \frac{-2}{\sqrt{6}}$$

$$\hat{n} = \frac{1}{\sqrt{6}}(1\hat{i} - 2\hat{j} + 1\hat{k})$$

This is the unit normal of the principal pla
direction cosines of principal plane correspon
to principal stress $\sigma_1 = 6$ KPa

$$n_x = \frac{1}{\sqrt{6}} \quad n_y = \frac{-2}{\sqrt{6}}$$

x For the stress tensor given below $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} N/m^2$

i Find stress invariants I_1, I_2 and I_3

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 3 + 0 = 3 \ N/m^2$$

$$I_2 = \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -4 - 1 - 1 = -6$$

$$I_3 = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} = 3(-4) - 1(-2) + 1(2)$$

$$= -12 + 2 + 2 = -8$$

ii Write down the charactristic equation for the stress tenson (cubic equ)

$$\sigma^3 - 3\sigma^2 - 6\sigma + 8 = 0$$

iii Determine the principal stress

solving the char. equation. We get principal Stress

$$\sigma_1 = -2 \quad \sigma_2 = 4 \quad \sigma_3 = 1$$

$$\sigma_1 = 4 \ N/m^2 \quad \sigma_2 = 4 \ N/m^2 \quad \sigma_3 = -2$$

iv Determine the unit normal

Principal plane corr. to the principal Stress $\sigma_1 = 4 \ N/m^2$

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = 0$$

$$\begin{bmatrix} 3-4 & 1 & 1 \\ 1 & 0-4 & 2 \\ 1 & 2 & 0-4 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -4 & 2 \\ 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = 0$$

$$-n_x + n_y + n_z = 0$$

$$n_x - 4n_y + 2n_z = 0$$

$$n_x + 2n_y - 4n_z = 0$$

$$n_x = n_y + n_z$$

$$n_x = 4n_y + 2n_z$$

$$0 = 3n_y - 3n_z$$

$$n_y = n_z$$

$$n_x + 2n_z - 4n_z = 0$$

$$n_x - 2n_z = 0$$

$$n_x = 2n_z$$

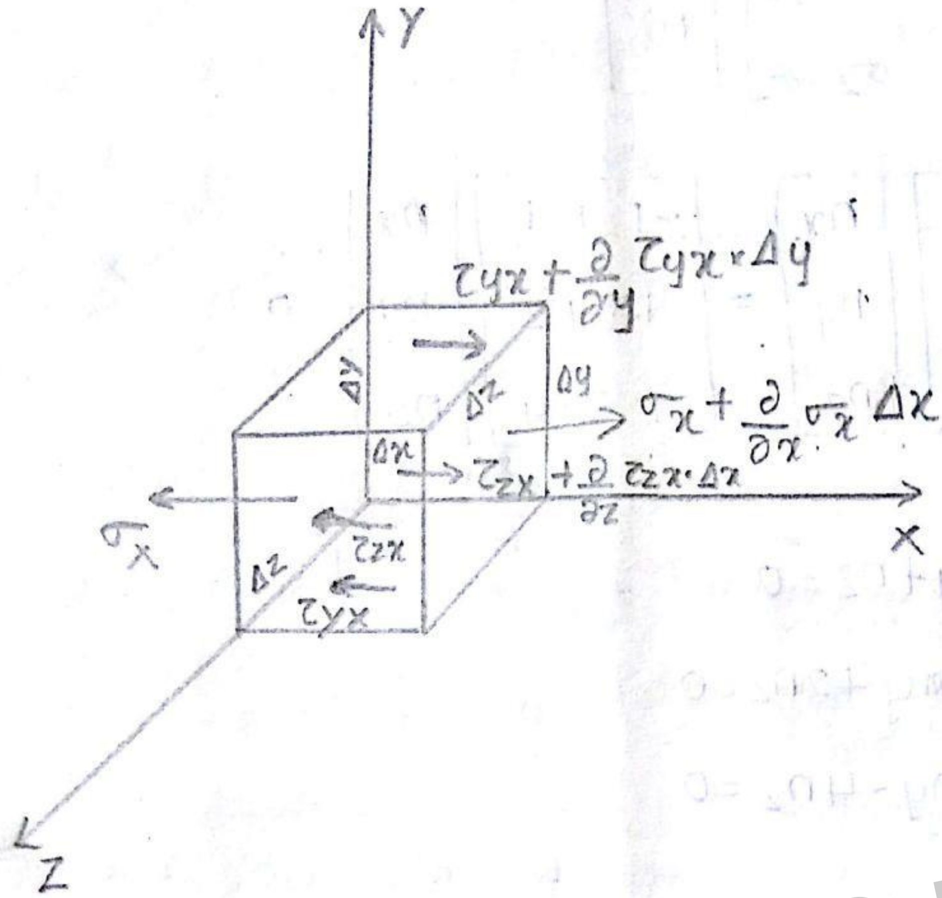
$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$4n_z^2 + n_z^2 + n_z^2 = 1$$

$$6n_z^2 = 1$$

$$n_z = \sqrt{1/6} \qquad n_y = \sqrt{1/6} \qquad n_x = \frac{2}{\sqrt{1/6}}$$

Equilibrium Equation in Cartesian co-ordinate



To find the force component in x direction we are considering the equilibrium condition

$$\Sigma F_x = 0$$

$$\left(\sigma_x + \frac{\partial}{\partial x}\sigma_x \cdot \Delta x\right)\underbrace{\Delta y \Delta z}_{\text{stress} \times \text{area}} - \sigma_x \Delta y \Delta z + \left(\tau_{yx} + \frac{\partial}{\partial y}\tau_{yx}\Delta y\right)\Delta x \Delta z$$

$$- \tau_{yx}\Delta x \Delta z + \left(\tau_{zx} + \frac{\partial}{\partial z}\tau_{zx}\Delta z\right)\Delta x \Delta y - \tau_{zx}\Delta x \Delta y + f(x) = 0$$

$$\cancel{\sigma_x \Delta y \Delta z} + \frac{\partial}{\partial x}\sigma_x \Delta x \Delta y \Delta z - \cancel{\sigma_x \Delta y \Delta z} + \tau_{yx}\Delta x \Delta z +$$

$$\frac{\partial}{\partial y}\Delta y \Delta x \Delta z - \cancel{\tau_{yx}\Delta x \Delta z} + \cancel{\tau_{zx}\Delta x \Delta y} + \frac{\partial}{\partial z}\tau_{zx}\Delta z \Delta x \Delta y -$$

$$\tau_{zx}\Delta x \Delta y + f(x)\Delta x \Delta y \Delta z = 0$$

$$\frac{\partial}{\partial x}\sigma_x \Delta x\Delta y\Delta z+\frac{\partial}{\partial y}\tau_{yx}\Delta y\Delta x\Delta z+\frac{\partial}{\partial z}\tau_{zx}\Delta z\Delta x\Delta y=0$$

$$+ f(x)\Delta x\Delta y\Delta z$$

Body force in x direction per unit volume

all divided by $\Delta x\Delta y\Delta z$

$$\boxed{\frac{\partial}{\partial x}\sigma_x+\frac{\partial}{\partial y}\tau_{yx}+\frac{\partial}{\partial z}\tau_{zx}+f(x)=0}$$

Similary

$\Sigma F_y=0$

$$\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\sigma_y+\frac{\partial}{\partial z}\tau_{zy}+f(y)=0$$

Similary $\Sigma F_z=0$

$$\frac{\partial}{\partial x}\tau_{xz}+\frac{\partial}{\partial y}\tau_{yz}+\frac{\partial}{\partial z}\sigma_z+f(z)=0$$

The stress feed of a body is given by

$\sigma_{xx}=20x^2+y^2$

$\sigma_{yy}=30x^3+200$

$\sigma_{zz}=30x(y^2+z^2)$

$\tau_{xy}=\tau_{yx}=zx$

$\tau_{xz}=\tau_{zx}=y^2z$

$\tau_{yz}=\tau_{zy}=x^3y$

write down the corresponding stress tensor matrix.

$$\tau_{ij} = \begin{bmatrix} 20x^2+y^2 & zx & y^2z \\ zx & 30x^3+200 & x^3y \\ y^2z & x^3y & 30(y^2+z^2) \end{bmatrix}$$

b. Find out the components of body force. Required for satisfying the equilibrium of the body.

$$\frac{\partial}{\partial x}\sigma_x + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} + f(x) = 0$$

$$\frac{\partial}{\partial x}(20x^2+y^2) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(y^2z) + f(x) = 0$$

$$20\times 2x + 0 + y^2 + f(x) = 0$$

$$f(x) = -40x - y^2 \quad (1)$$

$$\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\sigma_y + \frac{\partial}{\partial z}\tau_{zy} + f(y) = 0$$

$$\frac{\partial}{\partial x}zx + \frac{\partial}{\partial y}30x^2+200 + \frac{\partial}{\partial z}x^3y + f(y) = 0$$

$$z+0+0+f(y) = 0$$

$$f(y) = -z$$

$$\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\sigma_z = 0$$

$$\frac{\partial}{\partial x}y^2z + \frac{\partial}{\partial y}x^3y + \frac{\partial}{\partial z}30(y^2+z^2) + f(z) = 0$$

$$x^3 + 60z^2 + f(z) = 0$$

$$f(z) = -x^3 - 60z$$

Equlebrium Fo Equations

$$\frac{\partial \sigma}{\partial x} \quad \frac{\partial}{\partial x}(20x^2+y^2) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(y^2 z) - 40x - y^2$$

$$\frac{\partial}{\partial x}(zx) + \frac{\partial}{\partial y}(30x^2+200) + \frac{\partial}{\partial z} x^3 y - Z = 0$$

$$\frac{\partial}{\partial x} y^2 z + \frac{\partial}{\partial y} x^3 y + \frac{\partial}{\partial z} 30(x^2 y^2 + z^2) - x^3 - 60z = 0.$$

Hydraustatic and deviatoric stress

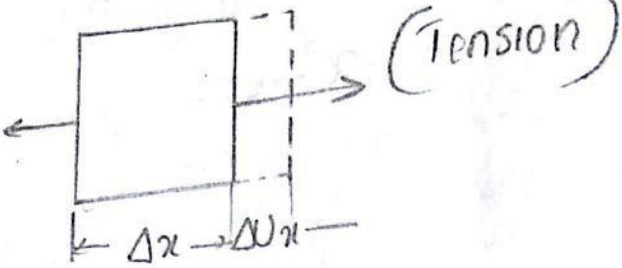
$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} + \begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{bmatrix}$$

stress tensor hydraustatic stress + Deviatoric stre:

$$I_1 = \sigma_x - \sigma + \sigma_y - \sigma + \sigma_z - \sigma$$

where $\sigma = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$

and when $I = 0$ Then it is called pure shear.



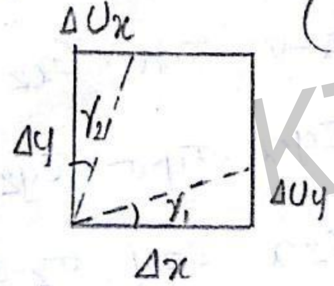
$$\varepsilon_x = \frac{\Delta U x}{\Delta x} \qquad \varepsilon_y = \frac{\Delta U y}{\Delta y} \qquad \varepsilon_z = \frac{\Delta U z}{\Delta z}$$

strain at a point $\lim_{\Delta x \to 0} \frac{\Delta U x}{\Delta x} = \frac{\partial U_x}{\partial x}$

$$\lim_{\Delta y \to 0} \frac{\Delta U y}{\Delta y} = \frac{\partial U_y}{\partial y}$$

$$\lim_{\Delta z \to 0} \frac{\Delta U z}{\Delta z} = \frac{\partial U_z}{\partial z}$$

When (Shearing)



$$\tan \gamma_1 = \frac{\Delta U y}{\Delta x} = \gamma_1$$

$$\tan \gamma_2 = \frac{\Delta U x}{\Delta y} = \gamma_2$$

sum of γ_1 and γ_1 and values are same so

$$\text{strain Matrix } \varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \varepsilon_z \end{bmatrix}$$

The displacement of a body is given by

$\vec{U} = (x^2+y)\hat{i} + (3+z)\hat{j} + (x^2+2y)\hat{k}$ estimate for the point 1,2,3 on the original body for component of engineering strain.

$U_x = x^2+y$ $\qquad \varepsilon_x = \frac{\partial}{\partial x} U_x = \frac{\partial}{\partial x}(x^2+y) = 2x$

$U_y = 3+z$ $\qquad \varepsilon_y = \frac{\partial}{\partial y} U_y = \frac{\partial}{\partial y}(3+z) = 0$

$U_z = x^2+2y$ $\qquad \varepsilon_z = \frac{\partial}{\partial z} U_z = \frac{\partial}{\partial z}(x^2+2y) = 0$

$\cancel{\gamma_{xy} = \frac{\partial}{\partial x} U_y + \frac{\partial}{\partial y} U_x}$

$\cancel{= 0 + 2x = 2x}$

$\cancel{\frac{\gamma_{xz}}{2} = \frac{\partial}{\partial x} U_x + \frac{\partial}{\partial z} U_z}$

$\cancel{= 2x + 0 =}$

$\gamma_{xy} = \frac{\partial}{\partial x} U_y + \frac{\partial}{\partial y} U_x$

$= \frac{\partial}{\partial x}(3+z) + \frac{\partial}{\partial y}(x^2+y)$

$= 0+1 = \underline{\underline{1}}$

$\gamma_{xz} = \frac{\partial}{\partial x} U_z + \frac{\partial}{\partial z} U_x = \frac{\partial}{\partial x}(x^2+2y) + \frac{\partial}{\partial z}(x^2+y)$

$= 2x + 0$

$= \underline{\underline{2x}}$

$\gamma_{yz} = \frac{\partial}{\partial y} U_z + \frac{\partial}{\partial z} U_y = \frac{\partial}{\partial y}(x^2+2y) + \frac{\partial}{\partial z}(3+z)$

$= 2+1$

$= 3$

Finally matrix

$$= \begin{bmatrix} 2x & \frac{1}{2} & \frac{2x}{2} \\ \frac{1}{2} & 0 & \frac{3}{2} \\ \frac{2x}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

in question ~~its~~ it is given value of x, y, z as 1, 2, 3 so $x = 1$

$$= \begin{bmatrix} 2 & 1/2 & 1 \\ 1/2 & 0 & 3/2 \\ 1 & 3/2 & 0 \end{bmatrix}$$

Strain at a point

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Strain Analysis (compatibility conditions)

$$\varepsilon_x = \frac{\partial}{\partial x} u_x \quad \text{---(1)}$$

$$\varepsilon_y = \frac{\partial}{\partial y} u_y \quad \text{---(2)}$$

differentiating (1) w.r.t. y

$$\frac{\partial}{\partial y}\varepsilon_x = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} u_x\right)$$

$$= \frac{\partial^2 u_x}{\partial x \partial y} \quad \text{---(3)}$$

differentiate (3) w.r.t. y

$$\frac{\partial^2}{\partial y^2}\varepsilon_x = \frac{\partial}{\partial y}\left(\frac{\partial^2 u_x}{\partial x \partial y}\right) = \cancel{\frac{\partial^3 u_x}{\partial^2 x \partial y}}$$

$$= \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y}\right) \quad \text{---(a)}$$

Differentiate (2) w.r.t. x

$$\frac{\partial}{\partial x}\varepsilon_y = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} u_y\right)$$

$$= \frac{\partial^2 u_y}{\partial x \partial y} \quad \text{---(4)}$$

Differentiate (4) w.r.t. x

$$\frac{\partial^2}{\partial x^2}\varepsilon_y = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_y}{\partial x}\right) \quad \text{---(b)}$$

add (a) + (b)

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_y}{\partial x}\right)$$

$$= \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$= \frac{\partial^2}{\partial x \partial y}(\gamma_{xy})$$

$$\boxed{\frac{\partial^2}{\partial y^2}\varepsilon_x + \frac{\partial^2}{\partial x^2}\varepsilon_y = \frac{\partial^2}{\partial x \partial y}\gamma_{xy}}$$

similarly

$$\varepsilon_y = \frac{\partial}{\partial y} u_y \quad \text{—(1)} \qquad \varepsilon_z = \frac{\partial}{\partial z} u_z \quad \text{—(2)}$$

differentiating (1) w.r.t z

$$\frac{\partial}{\partial z}\varepsilon_y = \frac{\partial}{\partial z}\left(\frac{\partial u_y}{\partial y}\right) = \frac{\partial^2}{\partial z \partial y} u_y$$

$$\frac{\partial^2}{\partial z^2}\varepsilon_y = \frac{\partial^2}{\partial z \partial y}\left(\frac{\partial u_y}{\partial z}\right) \text{ —— (a)}$$

differentiating (2) w.r.t y

$$\frac{\partial}{\partial y}\varepsilon_z = \frac{\partial}{\partial y}\left(\frac{\partial u_z}{\partial z}\right) = \frac{\partial^2}{\partial y \partial z} u_z$$

$$\frac{\partial^2}{\partial y^2}\varepsilon_z = \frac{\partial^2}{\partial y \partial z} \cancel{u_z}\left(\frac{\partial u_z}{\partial y}\right) \text{ —— (b)}$$

adding (a) and (b)

$$\frac{\partial^2}{\partial z^2}\varepsilon_y + \frac{\partial^2}{\partial y^2}\varepsilon_z = \frac{\partial^2}{\partial y \partial z}\left(\frac{\partial u_y}{\partial z}\right) + \frac{\partial^2}{\partial y \partial z}\left(\frac{\partial}{\partial y}u_z\right)$$

$$= \frac{\partial^2}{\partial y \partial z}\left(\frac{\partial}{\partial z}u_y + \frac{\partial}{\partial y}u_z\right)$$

$$\boxed{\frac{\partial^2}{\partial z^2}\varepsilon_y + \frac{\partial^2}{\partial y^2}\varepsilon_z = \frac{\partial^2}{\partial y \partial z}\gamma_{yz}}$$

similarly

$$\varepsilon_z = \frac{\partial}{\partial z}u_z \quad \text{—(1)} \qquad \varepsilon_x = \frac{\partial}{\partial x}u_x \quad \text{—(2)}$$

differentiating —(1) 2 times w.r.t x

$$\cancel{\frac{\partial^2}{\partial z^2}} \quad \frac{\partial^2}{\partial x^2}\varepsilon_z = \frac{\partial^2}{\partial z \partial x}\left(\frac{\partial}{\partial x}u_z\right)$$

differentiating —(2) 2 times

$$\cancel{\frac{\partial^2}{\partial x^2}\varepsilon} \quad \frac{\partial^2}{\partial z^2}\varepsilon_x = \frac{\partial^2}{\partial x \partial z}\left(\frac{\partial}{\partial z}u_x\right)$$

$$\boxed{\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2}{\partial z^2}\varepsilon_x = \frac{\partial^2 \gamma_{zx}}{\partial x \partial z}}$$

Compatibility conditions

These are mathematical conditions to be satisfy by the ~~stress~~ are tensor in order to form a possible strain field.

- Determine ~~th~~ wheather the following strain field is possible.?

$$\varepsilon_x = 5 + x^2 + y^2 + x^4 + y^4$$

$$\varepsilon_y = 6 + 3x^2 + 3y^2 + x^4 + y^4$$

$$\gamma_{xy} \cancel{\varepsilon_z} = 10 + 4xy(x^2 + y^2 + 2)$$

~~These~~ The strain field given will be possible when the compatibility conditions are satisfied.

$$\frac{\partial^2}{\partial y^2}\varepsilon_x + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial}{\partial y}(5 + x^2 + y^2 + x^4 + y^4) + \frac{\partial}{\partial x}(6 + 3x^2 + 3y^2 + x^4 + y^4)$$

$$= \frac{\partial}{\partial y}(2y + 4y^3) + \frac{\partial}{\partial x}(6x + 4x^3)$$

$$= 2 + 12y^2 + 6 + 12x^2$$

$$= 12y^2 + 12x^2 + 8$$

$$= 12(x^2 + y^2) + 8$$

$$\cancel{\frac{\partial \cdot \gamma_{xy}}{\partial x \partial y}} \qquad \cancel{\frac{\partial}{\partial x}} \gamma_{xy} = 10 + 4x^3y + 4xy^3 + 8xy$$

$$= \frac{\partial}{\partial x}\gamma_{xy}$$

$$\frac{\partial^2}{\partial x}\gamma_{xy} = 4\times 3x^2 + 4y^3 + 8y$$

$$\frac{\partial^2}{\partial x^2}\gamma_{xy} = \cancel{12x^2 + 4y^3 + 8y} \quad 12x^2 + 12y^2 + 8$$

$$LHS = 12x^2 + 12y^2 + 8$$

$$RHS = \cancel{12x^2 + 4y^3 + 8y} \quad 12x^2 + 12y^2 + 8$$

$\therefore$ LHS = RHS

$\therefore$ LHS = RHS, the strain field is possible.

Compatibility conditions.

$$\gamma_{xy} = \frac{\partial}{\partial x}U_y + \frac{\partial}{\partial y}U_x \quad \text{—(1)}$$

$$\gamma_{yz} = \frac{\partial}{\partial y}U_z + \frac{\partial}{\partial z}U_y \quad \text{—(2)}$$

$$\gamma_{xz} = \frac{\partial}{\partial x}U_z + \frac{\partial}{\partial z}U_x \quad \text{—(3)}$$

differentiate (1), (2) and (3) w.r.t z, x, ar y respectively.

$$\frac{\partial}{\partial z}(\gamma_{xy}) = \frac{\partial^2}{\partial x \partial z}(U_y) + \frac{\partial^2}{\partial y \partial z}(U_x) \quad \text{—(4)}$$

$$\frac{\partial}{\partial x}\gamma_{yz} = \frac{\partial^2}{\partial x \partial y}U_z + \frac{\partial^2}{\partial x \partial z}U_y \quad \text{—(5)}$$

$$\frac{\partial}{\partial y}\gamma_{xz} = \frac{\partial^2}{\partial x \partial y}U_z + \frac{\partial^2}{\partial z \partial y}U_x \quad \text{—(6)}$$

(6)+(5)−(4)

$$\frac{\partial}{\partial y}\gamma_{zx}+\frac{\partial}{\partial x}\gamma_{yz}-\frac{\partial}{\partial z}\gamma_{xy}=\frac{\partial^2}{\partial y\partial x}u_z+\frac{\partial^2}{\partial y\partial z}u_x+$$

$$\frac{\partial^2}{\partial x\partial z}u_y+\frac{\partial^2}{\partial x\partial y}u_z-\frac{\partial^2 u_x}{\partial z\partial y}-\frac{\partial^2 u_y}{\partial x\partial z}$$

$$=2\frac{\partial^2 u_z}{\partial y\partial x}$$

$$\frac{\partial}{\partial y}\gamma_{zx}+\frac{\partial}{\partial x}\gamma_{yz}-\frac{\partial}{\partial z}\gamma_{xy}=2\cdot\frac{\partial^2 u_z}{\partial y\partial x}$$

similarly

11)

Determine wheather the following strain field is possible.

$\cancel{\varepsilon_x = 5+}$

state the conditions under which following is a possible system of strain

$$\varepsilon_{xx}=a(x^2+y^2)+x^4+y^4$$

$$\varepsilon_{yy}=b(x^2+y^2)+x^4+y^4$$

$$\varepsilon_{zz}=0$$

$\gamma_{xy} = cxy(x^2+y^2+d^2)$

$\gamma_{yz} = 0$

$\gamma_{zx} = 0$

Write down the corresponding strain tensor.

strain tensor

$$\varepsilon_{ij} = \begin{bmatrix} a(x^2+y^2)+x^4+y^4 & \frac{1}{2}[cxy(x^2+y^2+d^2)] & 0 \\ \frac{1}{2}[cxy(x^2+y^2+d^2)] & b(x^2+y^2)+x^4+y^4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Determine the conditions

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\varepsilon_x = \frac{\partial}{\partial y}(a(x^2+y^2)+x^4+y^4)$$

$$= a \cdot 2y + 4y^3$$

$$\frac{\partial^2}{\partial y^2}\varepsilon_x = 2a + 12y^2$$

$$\frac{\partial^2}{\partial x^2}\varepsilon_y = \frac{\partial}{\partial x}\varepsilon_y = \frac{\partial}{\partial x}(b2y + 4x^3)$$

$$= 2b + 12x^2$$

$$\frac{\partial^2}{\partial x \partial y} \quad \frac{\partial \gamma_{xy}}{\partial x} = cy(\cancel{2x}$$

$$\gamma_{xy} = cxy(x^2+y^2+d^2)$$
$$= cx^3y + cxy^3 + cxyd^2$$
$$\frac{\partial \gamma_{xy}}{\partial x} = cy \times 3x^2 + cy^3 + cyd^2$$
$$\frac{\partial \gamma_{xy}}{\partial y} = cx^3 + cx3y^2 + cxd^2 \quad = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(cxy(x^2+y^2+d^2))\right)$$
$$\frac{\partial}{\partial x \partial y}\gamma_{xy} = 3x^2c + 3y^2c + cd^2$$
$$= c(3x^2+3y^2+d^2)$$

$$2a+12y^2+2b+12x^2 = c(3x^2+3y^2+d^2)$$

comparing co-efficients of x^2

$$12x^2 = c3x^2$$
$$c = \frac{12}{3} = 4$$

Comparing constants.

$$2a+2b = cd^2$$
$$2a+2b = 4d^2$$
$$a+b = 2d^2$$

when $c=4$ and $a+b=2d^2$ then only the strain field is possible.

Determine wheather the following strain field is possible.

$$\varepsilon_x = 2x^2 + 3y^2 + z + 1$$

$$\varepsilon_y = x^2 + 2y^2 + 3z + 2$$

$$\varepsilon_z = 3x + 2y + z^2 + 1$$

$$\gamma_{xy} = 8xy$$

$$\gamma_{xz} = \gamma_{yz} = 0$$

$$\varepsilon_{ij} = \begin{bmatrix} 2x^2+3y^2+z+1 & \frac{1}{2}8xy & 0 \\ \frac{1}{2}8xy & x^2+2y^2+3z+2 & 0 \\ 0 & 0 & 3x+2y+z^2+1 \end{bmatrix}$$

strain tensor

$$\frac{\partial^2}{\partial y^2}\varepsilon_x + \frac{\partial^2}{\partial x^2}\varepsilon_y = \frac{\partial^2}{\partial x \partial y}\gamma_{xy}$$

$$\frac{\partial}{\partial y}(2x^2+3y^2+z+1) = \frac{\partial^2}{\partial y^2}(6y) = 6$$

$$\frac{\partial}{\partial x}(x^2+2y^2+3z+2) = \frac{\partial^2}{\partial x^2}(2x) = 2$$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}(8xy)\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}8x\right]$$

$$\frac{\partial^2}{\partial x \partial y}\gamma_{xy} = \underline{\underline{8}}$$

$$\frac{\partial^2}{\partial y^2}\varepsilon_x + \frac{\partial^2}{\partial x^2}\varepsilon_y = 6+2 = 8 = \frac{\partial^2}{\partial x \partial y}\gamma_{xy}$$

LHS = RHS compatibility conditions are satisfi

The first matrix on the Right hand side has the same value of principal stress $= \sigma$.
Corresponding to this matrix, all the planes passing through the respective point will carry the same normal stress $= \sigma$ and zero shear stress. This characteristic ~~is identical~~ is identical to the hydrostatic Pr. ~~stress~~ in a static fluid. Hence it is called hydrostatic stress.

First invariant of the 2nd stress matrix on the R.H.S of the above equation.

$$I_1 = \sigma_x - \sigma + \sigma_y - \sigma + \sigma_z - \sigma$$

$$= (\sigma_x + \sigma_y + \sigma_z) - 3\sigma$$

if we take $\sigma = \dfrac{\sigma_x + \sigma_y + \sigma_z}{3}$

Then $I_1 = 0$ mean

This state of shear is called pure shear

The second composed matrix

$\sigma = \dfrac{\sigma_x + \sigma_y + \sigma_z}{3}$ is said to have pure shear that is called deviatoric stress

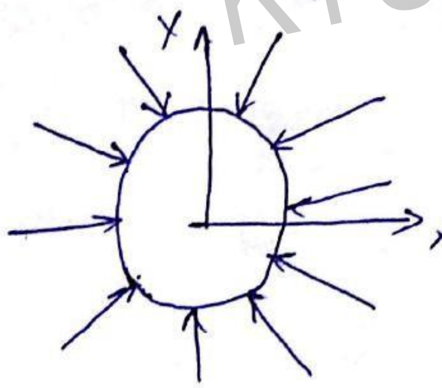
MODULE - II

2D problems in Elasticity.

2D problems in Elasticity are plane stress, plain strain and axisymmetric.

3x3 matrix representing stress and strain at a point of 3D problems is simplify to a 2 dimensional problem. This problems are defined in a region over a plane.

$$2D\ Problem = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$



Thin plate subjected to external loading (along xy plane)

$\sigma_z = 0$

$\tau_{xz} = 0$

$\tau_{yz} = 0$

} This is called plane stress condition.

Equilibrium equation corresponding to plane stress condition.

$$\frac{\partial}{\partial x}\sigma_x + \frac{\partial}{\partial y}\tau_{yx} + f(x) = 0$$

$$\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\sigma_y + f(y) = 0.$$

When equilibrium equations are possible (satisfied) then the strain field is possible.

Plain strain.

$\varepsilon_z = 0$

$\gamma_{xz} = 0$

$\gamma_{yz} = 0$

This condition is possible when the dimension on the solid is very large in z direction compared to x and z direction.

1/3/2017

Airy's stress function

Airy's stress function ϕ. It is theoretically ~~Possib~~ possible to determine a function satisfying the equilibrium equation, compactibility equation and boundary conditions.

In A 2 dimensional problem, ... a function, called aery's stress function denoted by ϕ. Which is a function of x and y

For this function, we assume that weight of body is only body force.

Let there exist a body force potential $V = \rho g$ such that

$$F(x) = \frac{\partial V}{\partial x}$$

$$F(y) = \frac{\partial V}{\partial y}$$

and defines $\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2}$$

$$\tau_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y}$$

This function ϕ will satisfy the equilibrium equation and compactibility equation with the condition

$\nabla^4 \phi = 0$ (ϕ is bi-harmonic function)

If the body forces are zero

$$\nabla^4 \phi$$

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

eg: $\phi = ax^2 + bxy + cy^2$

where

a, b, c are constants.

determine stress components

σ_x, σ_y and σ_z

$$\sigma_x = \frac{\partial}{\partial y}(ax^2+bxy+cy^2)$$

$$= bx+2cy$$

$$= \frac{\partial^2}{\partial y^2}(bx+2cy) = 2c \qquad \frac{\partial^3\phi}{\partial y^3} = 0$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} \qquad \frac{\partial^4\phi}{\partial y^4} = 0$$

$$= \frac{\partial}{\partial x}(ax^2+bxy+cy^2)$$

$$= 2ax+by$$

$$= \frac{\partial^2}{\partial x^2}(2ax+by) = 2a \qquad \frac{\partial^3\phi}{\partial x^2} = 0 \qquad \frac{\partial^4\phi}{\partial x^4} = 0$$

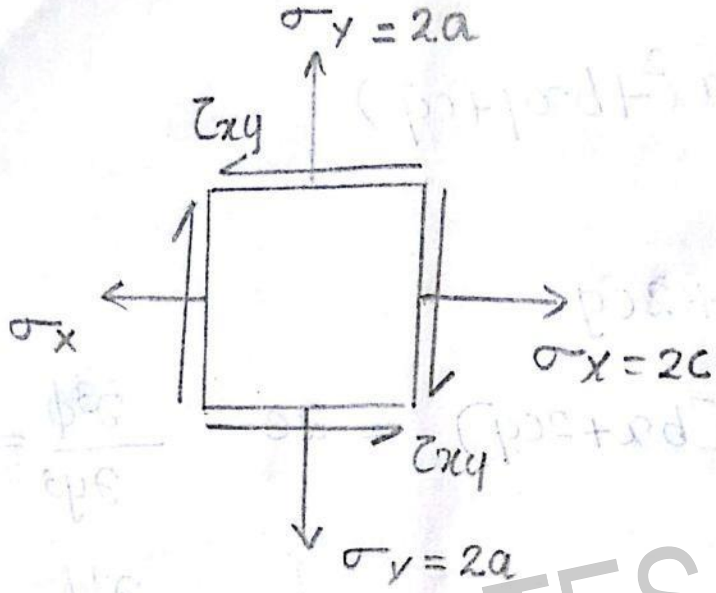
$$\tau_{xy} = \frac{-\partial^2\phi}{\partial x \partial y}$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} ax^2+bxy+cy^2\right)$$

$$= -\frac{\partial}{\partial x}(bx+2cy) \qquad \nabla^4\phi = 0 \quad \frac{\partial^4\phi}{\partial y^4} + \frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4}{\partial x^2 \ldots} = 0.$$

$$\text{stress tensor} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

$$\tau_{ij} = \begin{bmatrix} 2c & -b \\ -b & 2a \end{bmatrix}$$



1/4 Generalised hooke's law

$$\sigma_{xx} = a_{11}\varepsilon_{xx} + a_{12}\varepsilon_{yy} + a_{13}\varepsilon_{zz} + a_{14}\gamma_{xy} + a_{15}\gamma_{yz} + a_{16}\gamma_{zx}$$

for isotropic material constants are same

$$a_{11} = a_{12} = a_{13} = a_{14} = a_{15} = a_{16}$$

$\cancel{\sigma_{yy} = a_{11}\varepsilon}$

$$\sigma_{yy} = a_{21}\varepsilon_{xx} + a_{22}\varepsilon_{yy} + a_{23}\varepsilon_{zz} + a_{24}\gamma_{xy} + a_{25}\gamma_{yz} + a_{26}\gamma_{zx}$$

and similarly

$$\sigma_{zz} = a_{31}\varepsilon_{xx} + a_{32}\varepsilon_{yy} + a_{33}\varepsilon_{zz} + a_{34}\gamma_{xy} + a_{35}\gamma_{yz} + a_{36}\gamma_{zx}$$

$$\tau_{xy} = a_{41}\varepsilon_{xx} + a_{42}\varepsilon_{yy} + a_{43}\varepsilon_{zz} + a_{44}\gamma_{xy} + a_{45}\gamma_{yz} + a_{46}\gamma_{zx}$$

$$\tau_{xyz} = a_{51}\varepsilon_{xx} + a_{52}\varepsilon_{yy} + a_{53}\varepsilon_{zz} + a_{54}\gamma_{xy} + a_{55}\gamma_{yz} + a_{56}\gamma_{zx}$$

$$\tau_{zx} = a_{61}\varepsilon_{xx} + a_{62}\varepsilon_{yy} + a_{63}\varepsilon_{zz} + a_{64}\gamma_{xy} + a_{65}\gamma_{yz} + a_{66}\gamma_{zx}$$

componets of stress acting on an elastic body are related to components of strain.

The equations relating the components of stress and strain are called constitutive equations.

constitutive equations contaens co-efficcents related to elastic behaviour of the material of the body and are usually ~~interveing~~ determined by testing of material.

Since there are 6 components each for stress and strain, the most general form of the equation consists of 36 material co-efficients.

For homogeneous linear elastic materials the co-efficients $a_{11}, a_{12}, a_{13} \ldots\ldots a_{66}$ are constants

The above relations constant values for the co-efficients are called generalised hook's law

Stress-strain Relations of Isotropic Material.

$$\sigma_{xx} = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

λ = lamies co-efficient

Similarly

$$\sigma_{yy} = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\sigma_{zz} = 2G\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\lambda = \frac{\mu E}{(1-2\mu)(1+\mu)}$$

For a given state of strain at a point

$\varepsilon_x = 0.01$ $\varepsilon_y = 0.02$ $\varepsilon_z = -0.03$ $\gamma_{xy} = 0.001$

$\gamma_{yz} = 0$ $\gamma_{zx} = 0$

Determine stress components at the point

$E = 2.1 \times 10^6$ kg/cm^2

$G = 0.78 \times 10^6$ kg/cm^2

$\mu = 0.3$

$$\sigma_{xx} = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\lambda = \frac{\mu E}{(1-2\mu)(1+\mu)}$$

$$= \frac{0.3 \times 2.1 \times 10^6}{(1-2\times 0.3)(1+0.3)} = \underline{\underline{1211538.4}}$$

$$= 1.2115 \times 10^6 \text{ kg/cm}^2$$

$$\sigma_{xx} = 2\times 0.78\times10^{6}\times 0.01 + 1.2115\times10^{6}(0.1+0.2-0.3)$$

$$= \underline{\underline{15600\cancel{0}}}\ N/cm^2 \text{ or } kgF/cm^2$$

$$\sigma_{yy} = 2\times 0.78\times10^{\cancel{6}6}\times 0.02$$

$$= \underline{\underline{31200\cancel{0}}}\ N/cm^2$$

$$\sigma_{zz} = 2\times0.78\times10^{6}\times -0.03$$

$$= \underline{\underline{-46800\cancel{0}}}\ N/cm^2$$

$\cancel{\tau_{xy} = G}$

$$G = \frac{\tau_{xy}}{\gamma_{xy}}$$

$$\tau_{xy} = G\times\gamma_{xy}$$

$$= 0.78\times10^{6}\times 0.001$$

kgf/cm^2

$$= \underline{\underline{780}}\ kgf/cm^2$$

$$G = \frac{\tau_{yz}}{\gamma_{yz}} \qquad \left.\begin{array}{l}\tau_{yz}=0\\ \tau_{zx}=0\end{array}\right\}\ \text{as } \begin{array}{l}\gamma_{yz}=0\\ \gamma_{zx}=0\end{array}$$

$$\text{Strain tensor} = \begin{bmatrix} 0.01 & \frac{0.001}{2} & 0 \\ \frac{0.001}{2} & 0.02 & 0 \\ 0 & 0 & -0.03 \end{bmatrix}$$

Then the corresponding stress tensor

$$\Longrightarrow \begin{bmatrix} 15600 & 780 & 0 \\ 780 & 31200 & 0 \\ 0 & 0 & -46800 \end{bmatrix} kgf/cm^2$$

stress components are given

$\sigma_x = 15600$ ~~kgf/m^2~~ $\sigma_y = 31200$ ~~N/m^2~~ $\sigma_z = -46800$ ~~N/m^2~~

all are in kgf/cm^2 $\tau_{xy} = 780\ kgf/cm^2$ $\tau_{xz} = \tau_{yz} = 0$

$\mu = 0.3$ $E = 2.1 \times 10^6\ kg/cm^2$ $G = 0.78 \times 10^6\ kg/cm^2$

Find out the corresponding strain components.

$$\varepsilon_x = \frac{\sigma_x}{E} - \mu \cdot \frac{\sigma_y}{E} - \mu \cdot \frac{\sigma_z}{E}$$

$$= \frac{15600}{2.1 \times 10^6} - 0.3 \times \frac{31200}{2.1 \times 10^6} - 0.3 \times \frac{-46800}{2.1 \times 10^6}$$

$$= \underline{\underline{0.009657}}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \mu \cdot \frac{\sigma_x}{E} - \mu \cdot \frac{\sigma_z}{E}$$

$$= \frac{31200}{2.1 \times 10^6} - 0.3 \times \frac{15600}{2.1 \times 10^6} - 0.3 \times \frac{-46800}{2.1 \times 10^6}$$

$$= \cancel{0.019314}\ \cancel{5.9428 \times 10^{-3}}\ \underline{0.019314}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \mu \cdot \frac{\sigma_x}{E} - \mu \cdot \frac{\sigma_y}{E}$$

$$= \frac{-46800}{2.1\times10^6} - 0.3\times\frac{15600}{2.1\times10^6} - 0.3\times\frac{31200}{2.1\times10^6}$$

$$= \underline{\underline{-0.028971}}$$

$$G = \frac{\tau_{xy}}{\gamma_{xy}}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{780}{0.78\times10^6} = 0.001$$

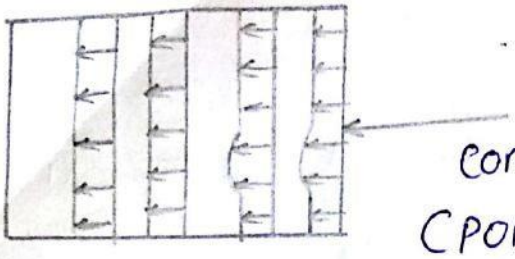
$$\gamma_{xz} = 0$$

$$\gamma_{yz} = 0$$

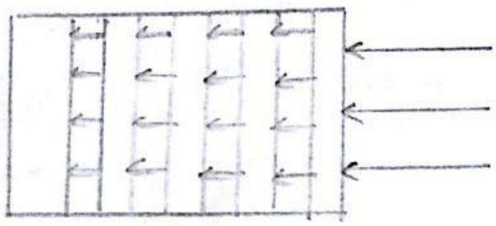
17

Saint venants principle:

It provides methodology for extending applicatio of the boundary conditions to continuous load as well.



concentratid load
(Point load is applying)



uniform traction.

Replacment of a concentrated load into a uniform traction load.

Saint venants principle

It states that a charge of loading distribution by a statically equivalent system of force having the same result force and couple on a small part of the surface of the body would give rise to locaused changes in stress and strain only. sufficiently away from the area stress and strain field will not be affected.

Uniqueness theorem

Any physically realistic elasticity problem defined by a set of governing equation and set of boundary conditions will have one and only one solutions

Proof of uniqueness theorem

To prove the theorem, we assume that there are 2 stress tensor field σ and σ' satisfying the ~~equations~~ equilibrium equation, of boundary conditions.

σ

$$\frac{\partial}{\partial x}\sigma_x + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} + f(x) = 0$$

$$\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\sigma_y + \frac{\partial}{\partial z}\tau_{xy} + f(y) = 0$$

$$\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\sigma_z + f(z) = 0$$

Boundary conditions

$$\sigma_x \cdot n_x + \tau_{xy} n_y + \tau_{xz} \cdot n_z = F_x$$

$$\tau_{yx} n_x + \sigma_y n_y + \tau_{yz} n_z = F_y$$

$$\tau_{zx} \cdot n_x + \tau_{zy} n_y + \sigma_z n_z = F_z$$

σ'

$$\frac{\partial}{\partial x}\sigma_x' + \frac{\partial}{\partial y}\tau_{yx}' + \frac{\partial}{\partial z}\tau_{zx}' + f(x) = 0$$

$$\frac{\partial}{\partial x}\tau_{xy}' + \frac{\partial}{\partial y}\sigma_y' + \frac{\partial}{\partial z}\tau_{zy}' + f(y) = 0$$

$$\frac{\partial}{\partial x}\tau_{xz}' + \frac{\partial}{\partial y}\tau_{yz}' + \frac{\partial}{\partial z}\sigma_z' + f(z) = 0$$

$$\sigma_x' n_x + \tau_{xy}' n_y + \tau_{xz}' n_z = F_x$$

$$\tau_{yx}' n_x + \sigma_y' n_y + \tau_{yz}' n_z = F_y$$

$$\tau_{zx}' n_x + \tau_{zy}' n_y + \sigma_z' n_z = F_z$$

Subtracting equation (3) From (1)
Proof unique theorem

$$\frac{\partial}{\partial x}(\sigma_x - \sigma_x') + \frac{\partial}{\partial y}(\tau_{xy} - \tau_{yx}') + \frac{\partial}{\partial z}(\tau_{zx} - \tau_{zx}') = 0$$

$$\frac{\partial}{\partial x}(\tau_{xy} - \tau_{xy}') + \frac{\partial}{\partial y}(\sigma_y - \sigma_y') + \frac{\partial}{\partial z}(\tau_{zy} - \tau_{zy}') = 0$$

$$\frac{\partial}{\partial x}(\tau_{xz} - \tau_{xz}') + \frac{\partial}{\partial y}(\tau_{yz} - \tau_{yz}') + \frac{\partial}{\partial z}(\sigma_z - \sigma_z') = 0$$

1

Then subtracting (4) from 2

$$(\sigma_x - \sigma_x')n_x + (\tau_{xy} - \tau_{xy}')n_y + (\tau_{xz} - \tau_{xz}')n_z = 0$$

$$(\tau_{yx} - \tau_{yx}')n_x + (\sigma_y - \sigma_y')n_y + (\tau_{yz} - \tau_{yz}')n_z = 0$$

$$(\tau_{zx} - \tau_{zx}')n_x + (\tau_{zy} - \tau_{zy}')n_y + (\sigma_z - \sigma_z')n_z = 0$$

2

(1) equations show that the difference of the two stress field satisfies the equilibrium equations with zero body force.

In set (2) equation the difference of the two stress fields satisfy the condition of zero external reaction.

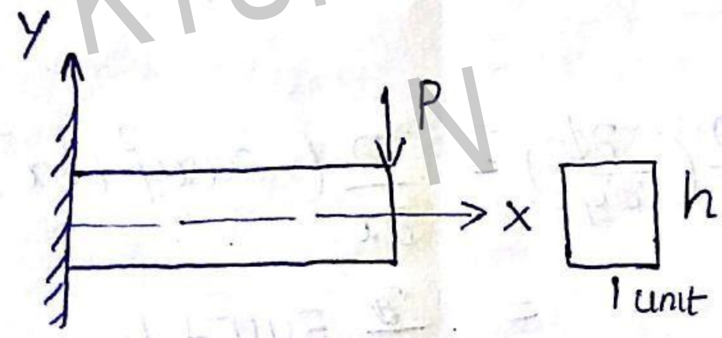
~~Strain field~~

Strain energy is proportional to the magnitude of external load applied.

The value of strain energy is with the difference of 2 stress field becomes equal to zero. Since strain energy is proportional to square of the stress at every point in the solid, the point vise difference of the 2 stress field is also zero.

Finally can be ~~concl~~ concluded that the assumptions of 2 stress fields satisfying the set of equation is not correct and there is one and only one stress field is as the solution.

$\phi = axy^3 + bxy$ be a stress function for a cantilever with concentrated load, at the end. Evaluate the constants a and b, if the load is P and the cross section is ~~ϕ~~ $1 \times h$



$$\nabla^4 \phi = 0.$$

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial y^4} + 2\frac{\partial^4 \phi}{\partial y^2 \partial x^2} = 0$$

$$\phi = axy^3 + bxy$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(axy^3 + bxy)$$

$$= (ay^3 + by)$$

$$= \frac{\partial^2 \phi}{\partial x^2} = 0 \quad \frac{\partial^4 \phi}{\partial x^4} = 0$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(axy^3 + bxy)$$

$$= \frac{\partial}{\partial y}(ax \cdot 3y^2 + bx)$$

$$\frac{\partial^2}{\partial y^2}(ax3y^2 + bx) = ax6 \cdot y$$

$$\frac{\partial^3}{\partial y^3}\phi = 6ax \quad \frac{\partial^4}{\partial y^4}\phi = 0$$

$$\frac{\partial \phi}{\partial y} \quad \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial x}(3axy^2 + bx)$$

$$= 3ay^2 + b$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \phi}{\partial y}\right) = \frac{\partial^2}{\partial x^2}(6ay)$$

$$= \underline{\underline{0}}$$

$$\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0.$$

$$\therefore \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial y^4} + 2 \cdot \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0.$$

$\therefore$ condition is satisfied.

$$\nabla^4 \phi = 0$$

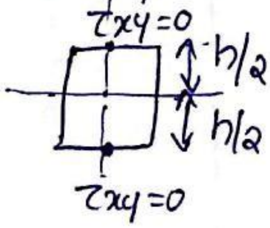
$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 6axy$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\tau_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y} = \frac{-\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}(axy^3 + bxy)\right)$$

$$= -[3ay^2 + b)$$

Arbitrary constants are evaluated by using boundary condition. The following boundary conditions are satisfied.



Apply boundary conditions to find a & b

$$\tau_{xy} = 0 \text{ at } y = \pm h/2$$

(at top and bottom position)

$$\tau_{xy} = -[3ay^2 + b] = 0$$

$$-\left[3a \times \left(\frac{h}{2}\right)^2 + b\right] = 0$$

$$-\left[\frac{3ah^2}{4} + b\right] = 0$$

$$\frac{3ah^2}{4} = +b$$

$$b = -\frac{3ah^2}{4}$$

Resultantant of distributed shear force at any cross section is equal to the applied load P.

$$\int_{-h/2}^{h/2} \tau_{xy}\, dA = 0.$$

$$\int_{-h/2}^{h/2} -[3ay^2+b]\, dA = 0$$

$$-\int_{-h/2}^{h/2} (3ay^2+b)\, dA = 0$$

dA is cross sectional area = dy×1

$$-\int_{-h/2}^{h/2} (3ay^2+b)\, dy = 0$$

$$-\left[3a \cdot \frac{y^3}{3} + 0\right]_{-h/2}^{h/2} = 0$$

$$-\left[3a \cdot 2x\frac{+h}{2} - 3a2\frac{-h}{2}\right] = 0$$

$$\left[3a \cdot \frac{y^3}{3} + by\right]_{-h/2}^{h/2} = 0$$

$$\frac{3a\left(\frac{h}{2}\right)^3}{3} + b \cdot \frac{h}{2} - \frac{3a\left(\frac{-h}{2}\right)^3}{3} + b\left(\frac{-h}{2}\right) = P$$

$$\frac{3ah^3}{24} + \frac{3ah^3}{24} = P$$

$$\frac{ah^3}{8} + \frac{ah^3}{8} = P$$

$$2 \cdot \frac{ah^3}{8} = P$$

$$\frac{ah^3}{4} = P$$

$$\frac{3ah^3}{24} + \frac{bh}{2} + \frac{3ah^3}{24} + \frac{bh}{2} = P$$

$$\frac{ah^3}{4} + bh = P$$

$$\frac{ah^3}{4} + \frac{-3ah^2}{4} \times h = P$$

$$\frac{ah^3}{4} - \frac{3}{4} ah^3 = P$$

$$\frac{ah^3 - 3ah^3}{4} = P$$

Investigate what problem of plane stress is satisfied by the stress function

$$\phi = \frac{3F}{4h}\left[xy - \frac{xy^3}{3h^2}\right] + \frac{P}{2} y^2$$

applied to the region $y=0$, $y=h$; $x=0$ on the side x +ve

[Stress function satisfies the biharmonic equation

$$\nabla^4\phi = 0$$

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{3F}{4h}\left[x - \frac{3xy^2}{3h^2}\right] + \frac{2Py}{2}$$

$$= \frac{3F}{4h}\left[x - \frac{xy^2}{h^2}\right] + Py$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{3F}{4h}\left[\frac{-2xy}{h^2}\right] + P \quad \text{——— (1)}$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2}$$

$$\frac{\partial\phi}{\partial x} = \frac{3F}{4h}\left[y - \frac{y^3}{3h^2}\right] + \frac{P}{2} \cdot 0$$

$$= \frac{3F}{4h}\left[y - \frac{y^3}{3h^2}\right]$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{3F}{4h}[0] = 0 \quad \text{——— (2)}$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{3F}{4h}\left[xy - \frac{xy^3}{3h^2}\right] + \frac{P}{2}y^2\right]\right.$$

$$= \frac{\partial}{\partial x}\left(\frac{3F}{4h}\left(x - \frac{3xy^2}{3h^2}\right) + \frac{1 \times P \times 2y}{2}\right)$$

$$= -\left[\frac{3F}{4h}\left[1 - \frac{y^2}{h^2}\right]\right] \quad \text{——— (3)}$$

$$\sigma_x = P - \frac{3F \times 2xy}{4h^3}$$

$$= P - \frac{1.5\,Fxy}{h^3}$$

$\sigma_y = 0$ for all values of x and y.

$$\tau_{xy} = -\left[\frac{3F}{4h} - \frac{3}{4}F\frac{y^2}{h^3}\right]$$

τ_{xy}, it varies parabolically with y

at $y=0$ $\tau_{xy} = \frac{-3F}{4h}$

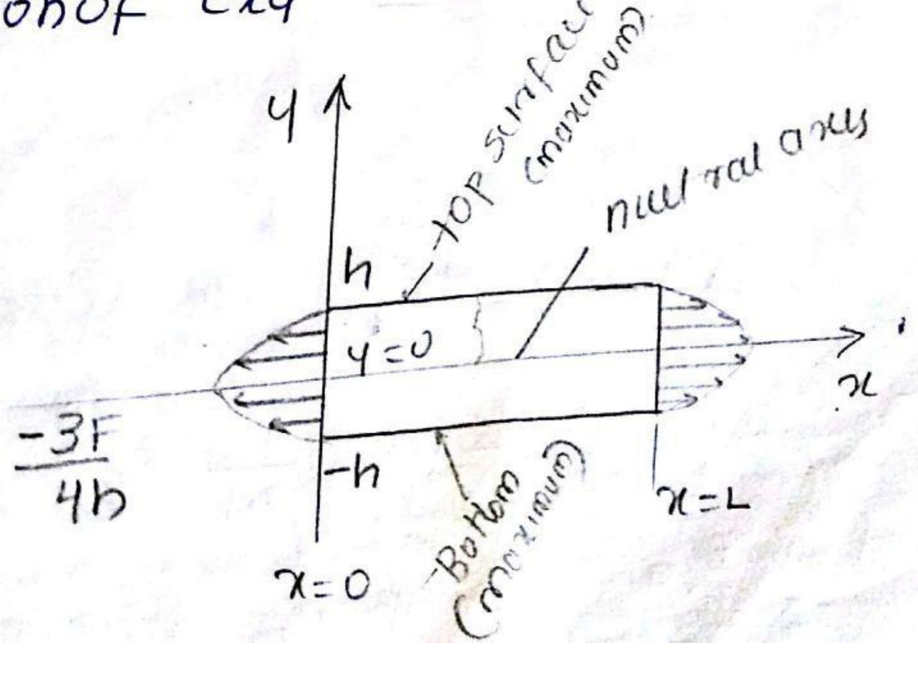
$y = \pm h$ $\tau_{xy} = -\left[\frac{3F}{4h} - \frac{3}{4}F\cdot\frac{h^2}{h^3}\right] = 0$

when $y = h$

$\tau_{xy} = -\left[\frac{3F}{4h} - \frac{3}{4}\frac{h^2}{h^3}\right] = -\left[\frac{2\cancel{3F}}{\cancel{4h}}\right]$ 0

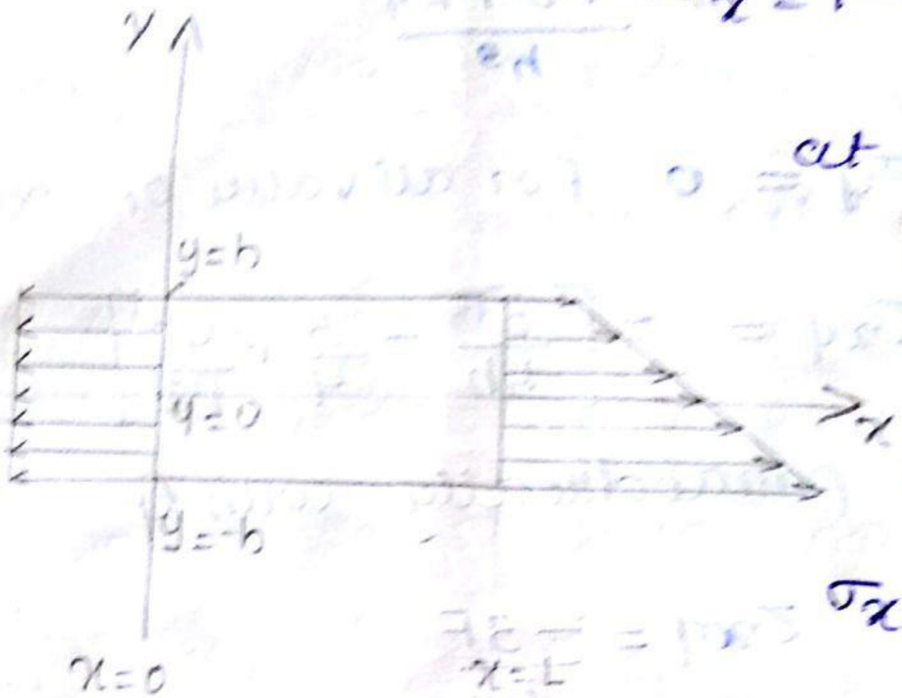
when $y = -h$ $= \cancel{\frac{-6F}{4h}}$

Variation of τ_{xy}



Variation of σ_x

$$\sigma_x = P - \frac{1.5\,Fxy}{h^3}$$



at $x=0$

$$\sigma_x = P$$

$$\sigma_x = P - \frac{1.5\,Fxy}{h^3}$$

at $x=L$

$$= P - \frac{1.5\,FLy}{h^3}$$

$$y = -h$$

$$\sigma_x = P + \frac{1.5\,FL}{h^2}$$

~~Variation of σ_y~~

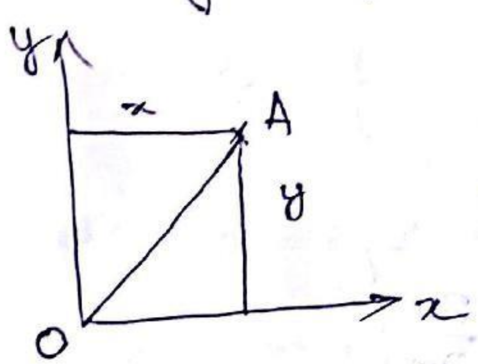
MODULE 3

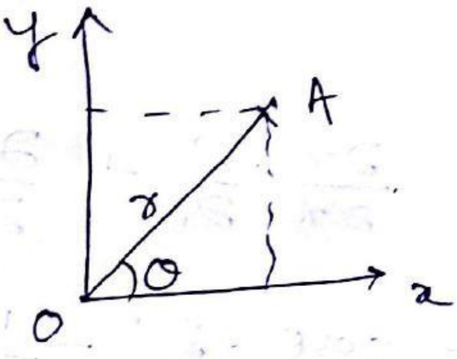
The cartesian or rectangular coordinate s/m studied so far is suited to body having straight or rectangular boundaries. However, for problems like cylinders, circular rings, disc, curved beams and plate containing holes etc., this coordinate s/m may not be suitable and hence a polar coordinate s/m need to be introduced.

Using polar coordinate s/m, hence the position of a point in the middle plane of a plate can be defined by the distance 'r' from the origin 'O' and by the angle 'θ' b/w 'r' and a certain axis OX on the fixed plane

Polar coordinate s/m

Consider the point 'A' having coordinates (x, y) in the rectangular s/m.





One can reach point 'A' by travelling a distance of

We have

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2+y^2}$$
and $\theta = \tan^{-1}(y/x)$

Now;
$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \times \cancel{2}x = \frac{x}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = -\frac{y}{r^2} = \frac{-y}{r} \cdot \frac{1}{r} = -\frac{1}{r}\sin\theta$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{x}{r^2} = \frac{1}{r} \cdot \frac{x}{r} = \frac{1}{r}\cos\theta$$

Now;
$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \cdot \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \cdot \frac{\partial}{\partial \theta}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x} = \cos\theta \cdot \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \cdot \frac{\partial}{\partial \theta}}$$

Similarly;
$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} \cdot \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \cdot \frac{\partial}{\partial \theta}$$

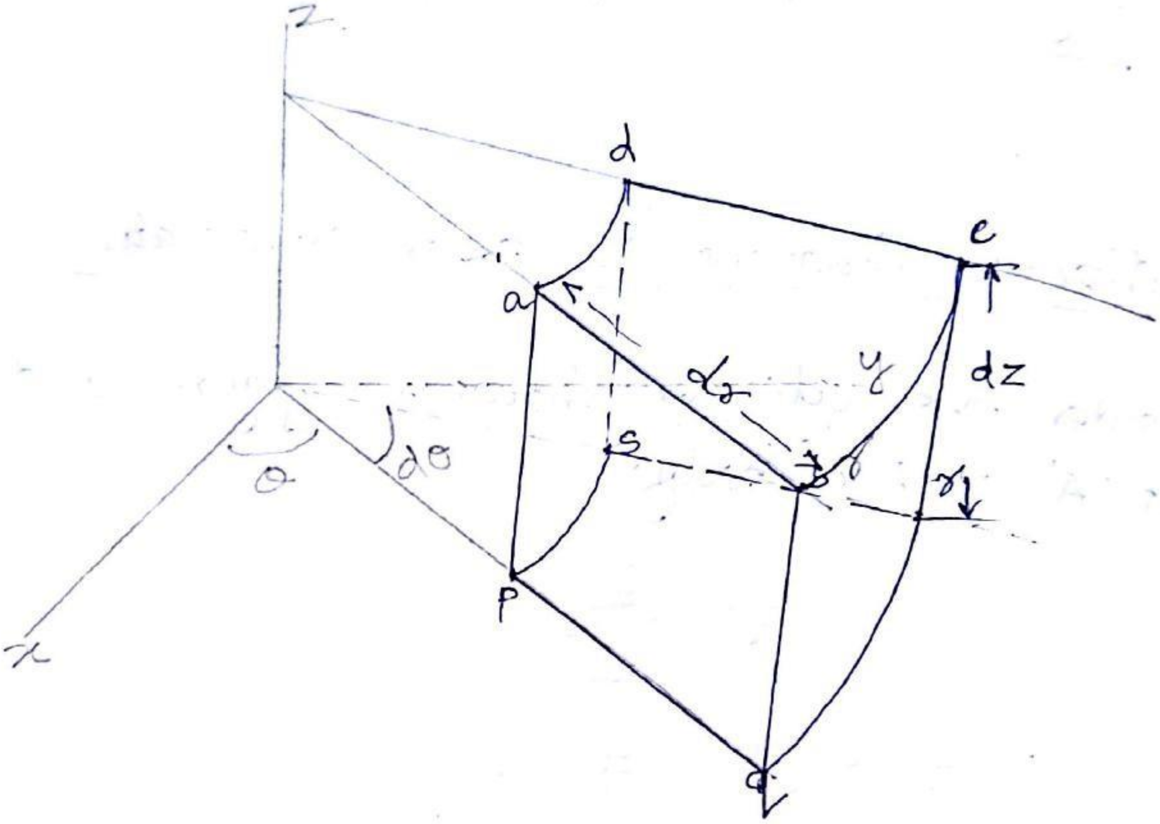
$$\Rightarrow \boxed{\frac{\partial}{\partial y} = \sin\theta \cdot \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \cdot \frac{\partial}{\partial \theta}}$$

Equilibrium Equations in polar Coordinates

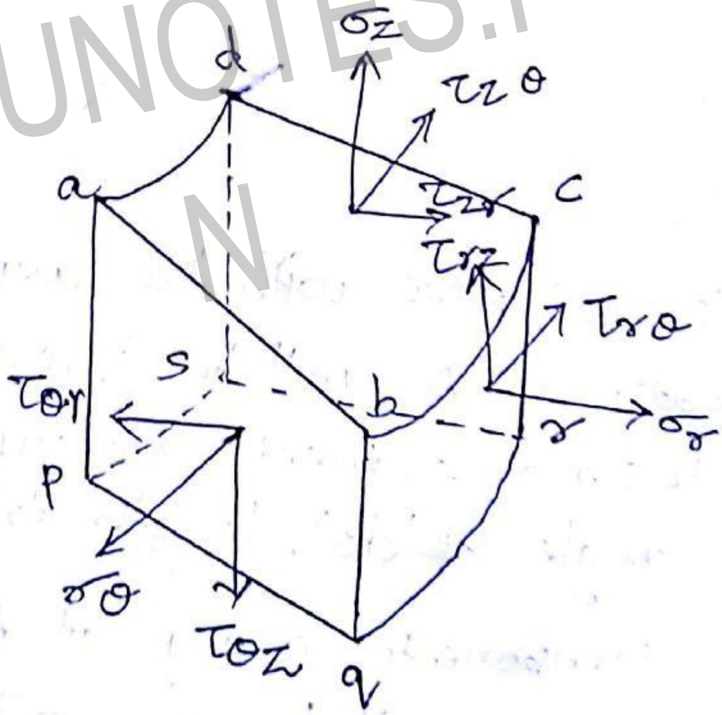
Consider the cylinder shown in figure and the point 'A' on the body.

The Z axis coincides with the axis of the cylinder. The coordinates of A will be (r, θ, z). Now consider an infinitesimal element surrounding 'A', with an included angle of '$d\theta$', length 'dr' and height 'dz'. The stress components acting at A along the coordinate axis will be σ_r, σ_θ, σ_z, $\tau_{r\theta}$, $\tau_{\theta z}$ and τ_{rz}.

P.T.O. →

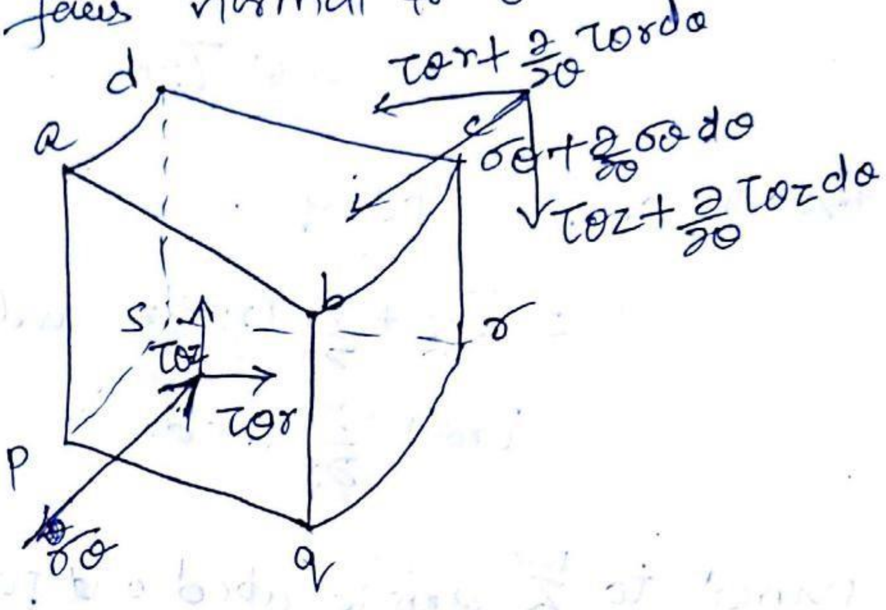


The stresses acting on this element can be shown as below;



Now; we can consider each area of this element separately and find the stress acting on ~~the~~ it

a) The faces normal to 'θ' i.e apqb and dcrs



Area of the face apqb $= dr \times dz$

lllly area of face dcrs $= dr \cdot dz$

Normal stress in the face apqb $= \sigma_\theta$

" " " " " dcrs $= \sigma_\theta + \frac{\partial}{\partial\theta}\sigma_\theta d\theta$

Shear stresses in the face apqb $= \tau_{\theta r}$ and $\tau_{\theta z}$

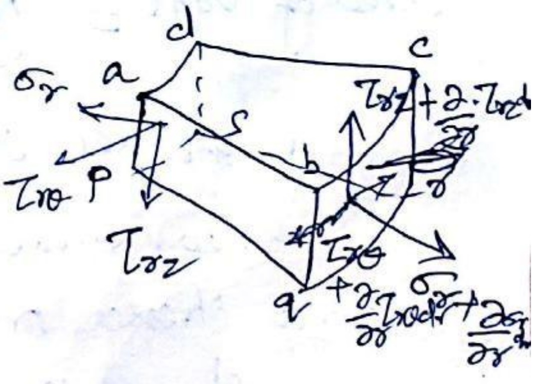
Shear " " " " dcrs $= \tau_{\theta r} + \frac{\partial}{\partial\theta}\tau_{\theta r} d\theta$

and $\tau_{\theta z} + \frac{\partial}{\partial\theta}\tau_{\theta z} d\theta$

b) faces normal to 'r' ie bcrq and adsp

area (bcrq) $= \cancel{r\,d\theta \cdot dz}\ (r+dr)\,d\theta \cdot dz$

Area (adsp) $= r\,d\theta\,dz$



Normal stresses:

on face bcrq

$= \sigma_r + \frac{\partial}{\partial r}\sigma_r dr$

on face adsp $= \sigma_r$

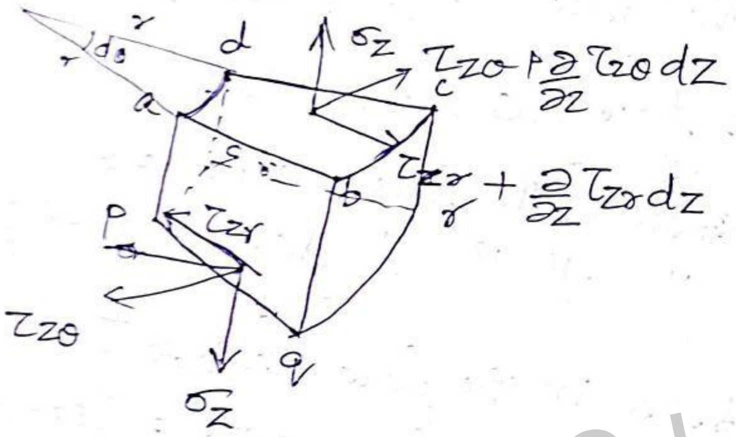
Shear stresses on face adsp
$= \tau_{rz}$ and $\tau_{r\theta}$

Shear stresses on face bcrq

$= \tau_{rz} + \frac{\partial}{\partial r}\tau_{rz}\,dr$ and

$\tau_{r\theta} + \frac{\partial}{\partial r}\tau_{r\theta}\,dr$

c) faces normal to 'z' axis ie abcd and pqrs



area of abcd $= \left(\frac{r\,d\theta + (r+dr)\,d\theta}{2}\right) dr$

$= \left(r + \frac{dr}{2}\right) d\theta\, dr$

area of pqrs $= \left(r + \frac{dr}{2}\right) d\theta . dr$

Normal stresses on face pqrs $= \sigma_z$

Shear stresses on face pqrs $= \tau_{zr}$ and $\tau_{z\theta}$

Normal stresses on face abcd $= \sigma_z + \frac{\partial}{\partial z}\sigma_z\,dz$

Shear stresses on face abcd $= \tau_{zr} + \frac{\partial}{\partial z}\tau_{zr}\,dz$

$\tau_{z\theta} + \frac{\partial}{\partial z}\tau_{z\theta}\,dz$

$ad = r\,d\theta$

$bc = (r+dr)\,d\theta$

$\left(r+\frac{dr}{2}\right)d\theta \times dr \times dz$

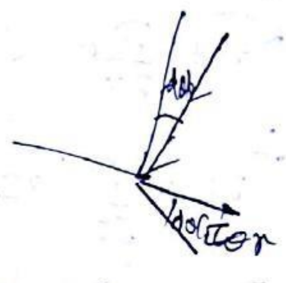
If the body force components are γ_r, γ_θ and γ_z

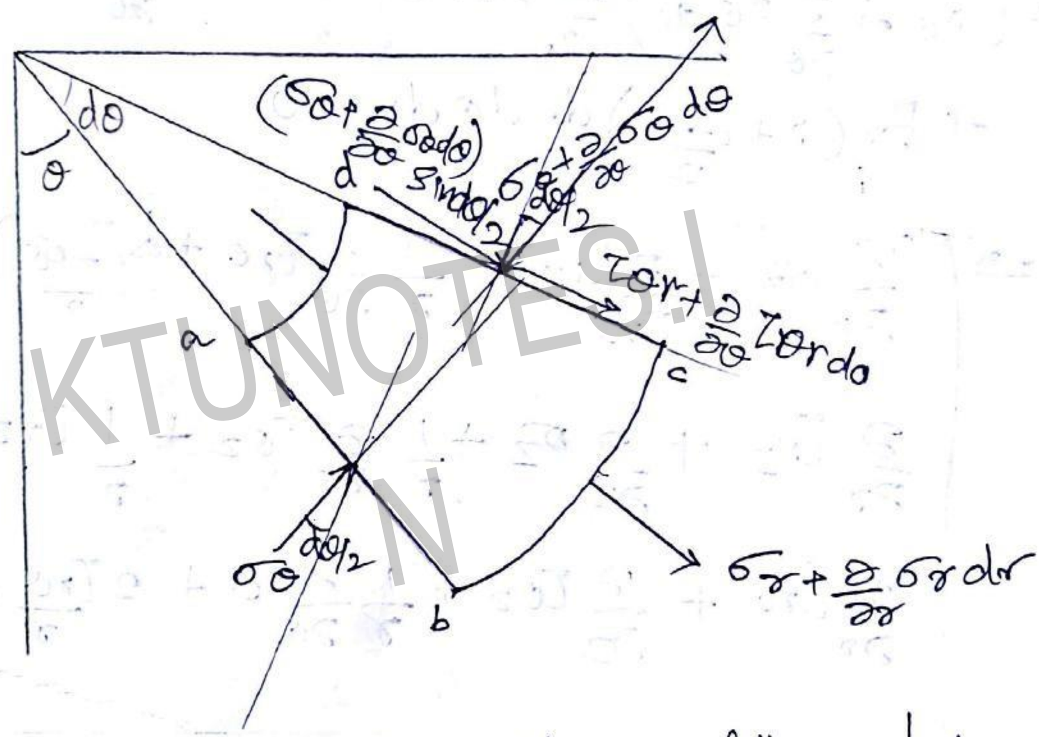
The force due to this will be

$$\gamma_r \left(r+\frac{dr}{2}\right)d\theta\cdot dr\cdot dz$$

$$\gamma_\theta \left(r+\frac{dr}{2}\right)d\theta\cdot dr\,dz$$

$$\gamma_z \left(r+\frac{dr}{2}\right)d\theta\cdot dr\cdot dz$$





Equilibrium equations can be written as follows. Let us first consider the equilibrium of forces along 'r'

$$\Rightarrow \left(\sigma_r+\frac{\partial}{\partial r}\sigma_r\,dr\right)(r+dr)\,d\theta\,dz - \sigma_r\,r\,d\theta\,dz$$

$$+\left(\tau_{zr}+\frac{\partial}{\partial z}\tau_{zr}\,dz\right)\left(r+\frac{dr}{2}\right)d\theta\,dz - \tau_{zr}\left(r+\frac{dr}{2}\right)d\theta\,dz$$

$$-\sigma_\theta \sin\frac{d\theta}{2}\,dr\,dz - \tau_{r\theta}\cos\frac{d\theta}{2}\,dr\,dz + \left(\sigma_\theta+\frac{\partial}{\partial\theta}\sigma_\theta\,d\theta\right)\sin\frac{d\theta}{2}\,dr\,dz$$

$$+\left(\tau_{r\theta}+\frac{\partial}{\partial\theta}\tau_{r\theta}\,d\theta\right)\cos\frac{d\theta}{2}\,dr\,dz + B_r\left(r+\frac{dr}{2}\right)dr\,d\theta\,dz = 0$$

$$\left.\begin{aligned} \sin\frac{d\theta}{2} &\approx \frac{d\theta}{2} \\ \text{and} \quad \cos\frac{d\theta}{2} &\approx 1 \end{aligned}\right\} \text{for small angles}$$

$$\left(\sigma_r + \frac{\partial}{\partial r}\sigma_r\,dr\right)(r+dr)\,d\theta\,dz + \left(\tau_{rz} + \frac{\partial}{\partial z}\tau_{rz}\,dz\right)\left(r+\frac{dr}{2}\right)d\theta\,dr - \sigma_r r\,d\theta\,dz$$
$$-\tau_{rz}\left(r+\frac{dr}{2}\right)d\theta\,dr - \sigma_\theta \cancel{\sin}\frac{d\theta}{2}\,dr\,dz - \tau_{r\theta}\cos\frac{d\theta}{2}\,dr\,dz$$
$$+\left(\sigma_\theta + \frac{\partial}{\partial\theta}\sigma_\theta\,d\theta\right)\frac{d\theta}{2}\,dr\,dz + \left(\tau_{r\theta} + \frac{\partial}{\partial\theta}\tau_{r\theta}\,d\theta\right)\cos\frac{d\theta}{2}\,dr\,dz$$
$$+B_r\left(r+\frac{dr}{2}\right)dr\,d\theta\,dz = 0$$

$$\Rightarrow \boxed{\begin{aligned} &\frac{\partial}{\partial r}\sigma_r + \frac{\partial}{\partial z}\tau_{rz} + \frac{1}{r}\frac{\partial}{\partial\theta}\tau_{r\theta} + \left(\frac{\sigma_r-\sigma_\theta}{r}\right) + B_r = 0 \\ &\frac{\partial}{\partial r}\tau_{rz} + \frac{\partial}{\partial z}\sigma_z + \frac{1}{r}\frac{\partial}{\partial\theta}\tau_{\theta z} + \frac{1}{r}\tau_{rz} + B_z = 0 \\ &\frac{\partial}{\partial r}\tau_{r\theta} + \frac{\partial}{\partial z}\tau_{\theta z} + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_\theta + 2\frac{\tau_{r\theta}}{r} + B_\theta = 0 \end{aligned}}$$

In 2D coordinates the above equation reduces to

$$\boxed{\begin{aligned} &\frac{\partial}{\partial r}\sigma_r + \frac{1}{r}\frac{\partial}{\partial\theta}\tau_{r\theta} + \left(\frac{\sigma_r-\sigma_\theta}{r}\right) + B_r = 0 \\ &\frac{\partial}{\partial r}\tau_{r\theta} + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_\theta + \frac{2\tau_{r\theta}}{r} + B_\theta = 0 \end{aligned}}$$

So, the above relation should be satisfied for ϕ as the stress function.

Please note that $\sigma_r = \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}$

and $\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$

Qn. Show that the function $\phi = Ar^4 \cos 4\theta + Br^4 \cos 2\theta$, qualifies as a stress function

$$\phi = Ar^4 \cos 4\theta + Br^4 \cos 2\theta$$

To qualify as a stress function; the following condition should be satisfied

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right)$$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = 12Br^2 \cos 2\theta$$

Now; $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(12Br^2 \cos 2\theta\right)$

$= 24B\cos 2\theta + 24B\cos 2\theta - 48B\cos 2\theta$

$= 0$

in polar coordinates

The components of engineering strain in polar coordinates are ε_{rr}, $\varepsilon_{\theta\theta}$ and $\tau_{r\theta}$

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

Strain tensor in polar coordinates

$$\begin{bmatrix} \varepsilon_{rr} & \frac{\gamma_{r\theta}}{2} & 0 \\ \frac{\gamma_{\theta r}}{2} & \varepsilon_{\theta\theta} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) & 0 \\ \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & 0 \\ 0 & 0 & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Stress strain relationships in polar coordinates

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right]$$

$$\gamma_{r\theta} = \frac{2(1+\nu)}{E}\tau_{r\theta}$$

$$\gamma_{\theta z} = \frac{2(1+\nu)}{E}\tau_{\theta z}$$

$$\gamma_{zr} = \frac{2(1+\nu)}{E}\tau_{zr}$$

Airy's stress function in polar coordinates

In cartesian coordinates we have

$$\nabla^2(\sigma_x + \sigma_y) = 0$$

$$\nabla^2\left(\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2}\right) = 0$$

Here we have to find $\frac{\partial^2\phi}{\partial x^2}$ and $\frac{\partial^2\phi}{\partial y^2}$

Now, recall that; $x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

Hence $\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial\phi}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \quad ; \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial\theta}{\partial x} = -\frac{1}{r}\sin\theta \quad ; \quad \frac{\partial\theta}{\partial y} = \frac{1}{r}\cos\theta$$

We can rewrite the above equations as follows,

$$\frac{\partial \phi}{\partial x} = \cos\theta \cdot \frac{\partial \phi}{\partial r} - \frac{1}{r}\sin\theta \cdot \frac{\partial \phi}{\partial \theta}$$

$$\frac{\partial \phi}{\partial y} = \sin\theta \cdot \frac{\partial \phi}{\partial r} + \frac{1}{r}\cos\theta \cdot \frac{\partial \phi}{\partial \theta}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial x}\right]$$

$$= \left(\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \cdot \frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial \phi}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial \phi}{\partial \theta}\right)$$

$$= \cos\theta \frac{\partial}{\partial r}\left[\cos\theta \frac{\partial \phi}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial \phi}{\partial \theta}\right]$$

$$- \frac{1}{r}\sin\theta \cdot \frac{\partial}{\partial \theta}\left[\cos\theta \frac{\partial \phi}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial \phi}{\partial \theta}\right]$$

$$= \cos\theta\left[\frac{\partial^2 \phi}{\partial r^2}\cos\theta - \left(\frac{1}{r}\sin\theta \frac{\partial^2 \phi}{\partial r \partial \theta} + \left(-\frac{\sin\theta}{r^2}\right)\frac{\partial \phi}{\partial \theta}\right)\right]$$

$$- \frac{1}{r}\sin\theta\left[\left(\cos\theta \frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{\partial \phi}{\partial r}\sin\theta\right) - \left(\frac{1}{r}\sin\theta \cdot \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial \phi}{\partial \theta} \cdot \frac{1}{r}\cos\theta\right)\right]$$

$$= \frac{\partial^2 \phi}{\partial r^2}\cos^2\theta - \frac{1}{r}\sin\theta\cos\theta \frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial \phi}{\partial \theta}$$

$$- \frac{1}{r}\sin\theta\cos\theta \frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial \phi}{\partial r}\sin^2\theta + \frac{1}{r^2}\sin^2\theta \frac{\partial^2 \phi}{\partial \theta^2}$$

$$- \frac{1}{r^2}\sin\theta\cos\theta \, \partial\phi/\partial r$$

$$= \frac{\partial^2\phi}{\partial r^2}\cos^2\theta - \frac{2}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\cos\theta\sin\theta + \frac{1}{r}\frac{\partial\phi}{\partial r}\sin^2\theta + \frac{2}{r^2}\frac{\partial\phi}{\partial\theta}\sin\theta\cos\theta$$

Hence

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial r^2}\cos^2\theta - \frac{2}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\cos\theta\sin\theta + \frac{1}{r}\frac{\partial\phi}{\partial r}\sin^2\theta + \frac{2}{r^2}\frac{\partial\phi}{\partial\theta}\sin\theta\cos\theta$$

Now;

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial\phi}{\partial y}\right].$$

$$= \left[\sin\theta\cdot\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right]\left[\sin\theta\cdot\frac{\partial\phi}{\partial r} + \frac{1}{r}\cos\theta\cdot\frac{\partial\phi}{\partial\theta}\right]$$

$$= \sin\theta\cdot\frac{\partial}{\partial r}\left[\sin\theta\cdot\frac{\partial\phi}{\partial r} + \frac{1}{r}\cos\theta\cdot\frac{\partial\phi}{\partial\theta}\right] + \frac{1}{r}\cos\theta\,\frac{\partial}{\partial\theta}\left[\sin\theta\cdot\frac{\partial\phi}{\partial r} + \frac{1}{r}\cos\theta\cdot\frac{\partial\phi}{\partial\theta}\right]$$

$$= \sin\theta\cdot\left[\sin\theta\cdot\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\cos\theta\frac{\partial^2\phi}{\partial\theta^2} - \frac{1}{r^2}\cos\theta\cdot\frac{\partial\phi}{\partial\theta}\right]$$

$$+ \frac{1}{r}\cos\theta\left[\sin\theta\frac{\partial^2\phi}{\partial r\partial\theta} + \cos\theta\cdot\frac{\partial\phi}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial^2\phi}{\partial\theta^2} - \frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta}\right]$$

$$= \sin^2\theta\cdot\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\sin\theta\cdot\cos\theta\frac{\partial^2\phi}{\partial\theta^2} - \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial\phi}{\partial\theta}$$

$$+ \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{1}{r}\cos^2\theta\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\cos^2\theta\frac{\partial^2\phi}{\partial\theta^2} - \frac{1}{r^2}\sin\theta\cos\theta\,\partial\phi/\partial\theta$$

$$= \frac{\partial^2 \phi}{\partial r^2} \sin^2\theta + \frac{2}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} \sin\theta \cos\theta + \frac{1}{r} \frac{\partial \phi}{\partial r} \cos^2\theta$$
$$- \frac{2}{r^2} \frac{\partial \phi}{\partial \theta} \sin\theta \cos\theta + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \cos^2\theta$$

Hence $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$

$$= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$$

$$\nabla^4 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right)$$

This should be zero for ϕ to be used as the stress function

Note that

$$\sigma_r = \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r}$$

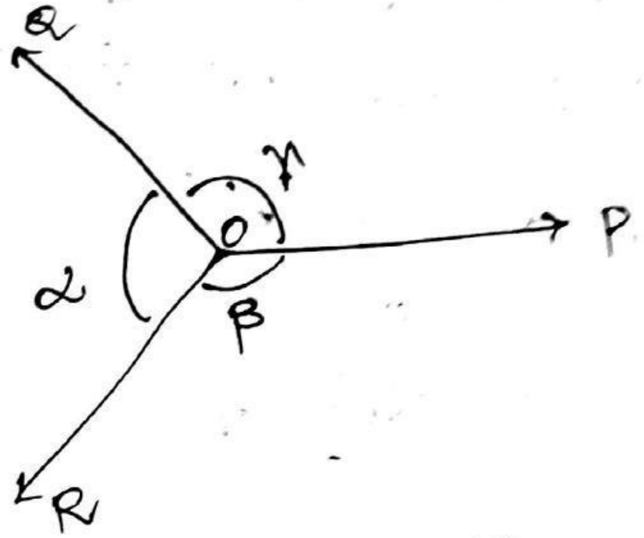
$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

$$\tau_{r\theta} = \left(\frac{1}{r^2}\right) \frac{\partial \phi}{\partial \theta} - \left(\frac{1}{r}\right) \frac{\partial^2 \phi}{\partial r \partial \theta}$$

are the stress components interms of Airy's stress function.

Lame's Theorem

If a body is in eqbm under the action of three forces, each force is proportional to the sine of the angle b/w the other two forces.



If P, Q and R are the three forces, then acc. to Lamé's theorem;

$$\boxed{\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma} = \text{Constant}}$$

Lame's Ellipsoid

Lame's stress ellipsoid is an alternative to Mohr's circle for the graphical representation of state of stress @ a point. The surface of the ellipsoid represent the end points of all the stress vectors ~~passing through~~ acting on all the planes passing through that point.

Once the equation of ellipsoid is known, the stress vector can be obtained for any plane passing through that point.

Inorder to calculate the equation of ellipsoid, consider the stress vector in terms of principal stresses,

$$\begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \\ \gamma \end{Bmatrix}$$

where $\alpha^2+\beta^2+\gamma^2 = 1$ is a unit vector

Thus,

$$\frac{f_x^2}{N_1^2} + \frac{f_y^2}{N_2^2} + \frac{f_z^2}{N_3^2} = 1$$

where $N_1 = \sigma_1$, $N_2 = \sigma_2$ and $N_3 = \sigma_3$

ie $$\frac{f_x^2}{\sigma_1^2} + \frac{f_y^2}{\sigma_2^2} + \frac{f_z^2}{\sigma_3^2} = 1$$

If $\sigma_1 = \sigma_2 = \sigma_3$, the ellipsoid becomes a sphere.

Note that, the magnitude of principal stress is the length of semi axes of the ellipsoid

Stress Concentration

Axisymmetric Problems

Axisymmetric problems are those in which the geometry as well as loading are axisymmetric. These problems are concerned with solid of revolution which are deformed symmetrically wrt axis of rotation. Hence, the deformation in this case will be symmetrical about the axis (say Z axis) and stress components are independent of 'θ' ie $\tau_{r\theta}$ and $\tau_{\theta z} = 0$

egs of axisymmetric problems are;

a. thick walled cylinder subjected to internal and external pressure

b. disk rotating about its axis

Thick cylinder subjected to internal and external Pressure

Remember the stress-strain relations

$$\varepsilon_r = \frac{\partial u_r}{\partial r} \quad u_r = \text{displacement along } r$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \quad u_\theta = \text{displacement along } \theta$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

Since the stress component in θ direction for an axisymmetric problem is zero; we have

$\cancel{\sigma_\theta = 0}$; $\tau_{r\theta} = 0$ and $\tau_{z\theta} = 0$

The equilibrium equations reduces to the following form:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + \gamma_r = 0$$

and

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial}{\partial r}\tau_{rz} + \frac{\tau_{rz}}{r} + \gamma_z = 0$$

Now; Consider a cylinder whose length is very large compared to the radius. Let the inner radius of the cylinder be 'a' and the outer radius be 'b'. The cylinder is subjected to an internal pressure P_a and an external pressure P_b.

Since the length of the cylinder is large compared to the radius, plane strain conditions exist. Hence, τ_{rz} in the equilibrium eqns become zero. Equilibrium equations reduces to the following form:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \gamma_r = 0$$

$$r \frac{d}{dr}\sigma_r + \frac{\sigma_r - \sigma_\theta}{r} + \gamma_r = 0$$

and.

$$\frac{d\sigma_z}{dz} + \gamma_z = 0$$

If there are no body forces, γ_r and γ_z are zero.

Then; $$\frac{d}{dr}\sigma_r + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\Rightarrow r\frac{d}{dr}\sigma_r + \sigma_r - \sigma_\theta = 0$$

$$\Rightarrow \boxed{\frac{d}{dr}(r\sigma_r) - \sigma_\theta = 0} \longrightarrow ①$$

from Hooke's law, we have;

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right] \longrightarrow ②$$

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] \longrightarrow ③$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right] \longrightarrow ④$$

for plane strain; $\varepsilon_z = 0$

$$\Rightarrow 0 = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right]$$

$$\Rightarrow \sigma_z = \nu(\sigma_r + \sigma_\theta)$$

Substituting σ_z in ② and ③

$$\Rightarrow \varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \nu\sigma_r + \nu\sigma_\theta)\right]$$

$$= \frac{1}{E}\left[\sigma_r - \nu\sigma_\theta - \nu^2\sigma_r - \nu^2\sigma_\theta\right]$$

$$= \frac{1}{E}\left[(1-\nu^2)\sigma_r - \nu(1+\nu)\sigma_\theta\right]$$

$$\Rightarrow \boxed{\varepsilon_r = \frac{1}{E}(1-\nu^2)\sigma_r - \nu(1+\nu)\sigma_\theta}$$

$$\boxed{\varepsilon_r = \frac{1}{E}(1+\nu)\left[(1-\nu)\sigma_r - \nu\sigma_\theta\right]} \longrightarrow (5)$$

Now;

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_z + \sigma_r)\right]$$

$$\Rightarrow \varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\nu\sigma_r + \nu\sigma_\theta + \sigma_r)\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu^2\sigma_r - \nu^2\sigma_\theta - \nu\sigma_r\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[(1-\nu^2)\sigma_\theta - \nu(1+\nu)\sigma_r\right]$$

$$\boxed{\varepsilon_\theta = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right]} \longrightarrow (6)$$

$$(1-\nu)\sigma_r - \nu\sigma_\theta = \frac{E\varepsilon_r}{(1+\nu)} \quad (\text{from } 5) \rightarrow (7)$$

$$(1-\nu)\sigma_\theta - \nu\sigma_r = \frac{E\varepsilon_\theta}{(1+\nu)} \quad (\text{from } 6) \rightarrow (8)$$

$$(7) \Rightarrow \nu\sigma_\theta = (1-\nu)\sigma_r - \frac{E\varepsilon_r}{(1+\nu)}$$

$$\Rightarrow \sigma_\theta = \frac{(1-\nu)}{\nu}\sigma_r - \frac{E}{\nu}\frac{\varepsilon_r}{(1+\nu)}$$

Substitute σ_θ in (8).

$$\Rightarrow (1-\nu)\left[\left(\frac{1-\nu}{\nu}\right)\sigma_r - \frac{E}{\nu}\frac{\varepsilon_r}{(1+\nu)}\right] - \nu\sigma_r = \frac{E\varepsilon_\theta}{(1+\nu)}$$

$$\frac{(1-\nu)^2\sigma_r}{\nu} - \frac{E}{\nu}\varepsilon_r\frac{(1-\nu)}{(1+\nu)} - \nu\sigma_r = \frac{E\varepsilon_\theta}{(1+\nu)}$$

$$\sigma_r\left[\frac{(1-\nu)^2}{\nu} - \nu\right] = \frac{E}{\nu}\varepsilon_r\frac{(1-\nu)}{(1+\nu)} + \frac{E\varepsilon_\theta}{(1+\nu)}$$

$$\sigma_r\left(\frac{1-2\nu+\cancel{\nu^2}-\cancel{\nu^2}}{\nu}\right) = \frac{E}{\nu}\varepsilon_r\frac{(1-\nu)}{(1+\nu)} + \frac{E\varepsilon_\theta}{(1+\nu)}$$

$$(1-2\nu)\sigma_r = E\left[\frac{(1-\nu)}{(1+\nu)}\varepsilon_r + \frac{\nu\varepsilon_\theta}{(1+\nu)}\right]$$

$$\boxed{\sigma_r = \frac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\varepsilon_r + \nu\varepsilon_\theta\right]} \longrightarrow (9)$$

similarly -

$$\boxed{\sigma_\theta = \frac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\varepsilon_\theta + \nu\varepsilon_r\right]} \longrightarrow (10)$$

However $\varepsilon_r = \frac{du_r}{dr}$

and $\varepsilon_\theta = \frac{u_r}{r}$ (since axisymmetric plane strain)

$\therefore$ (9) and (10) can be rewritten as follows:

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{du_r}{dr} + \nu\frac{u_r}{r}\right] \rightarrow (11)$$

$$\sigma_\theta = \frac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\frac{u_r}{r} + \nu\frac{du_r}{dr}\right] \rightarrow (12)$$

Substitute (11) and (12) in (1)

$$\Rightarrow \frac{d}{dr}(r\sigma_r) - \sigma_\theta = 0$$

$$\Rightarrow \frac{dr}{dr}\left[(1-\nu)\frac{du_r}{dr} + \nu\frac{u_r}{r}\right] - \left[(1-\nu)\frac{u_r}{r} + \nu\frac{du_r}{dr}\right] = 0$$

$$\Rightarrow \frac{d}{dr}\left[(1-\nu)r\frac{du_r}{dr} + \nu u_r\right] - \left[(1-\nu)\frac{u_r}{r} + \nu\frac{du_r}{dr}\right] = 0$$

$$\Rightarrow (1-\nu)\frac{du_r}{dr} + (1-\nu)r\frac{d^2u_r}{dr^2} + \cancel{\nu\frac{du_r}{dr}} - \cancel{\nu\frac{du_r}{dr}} - (1-\nu)\frac{u_r}{r} = 0$$

$$\Rightarrow \frac{du_r}{dr} + r\frac{d^2u_r}{dr^2} - \frac{u_r}{r} = 0$$

$$\cancel{\Rightarrow \frac{d}{dr}\left[\frac{du_r}{dr} + \frac{u_r}{r}\right] = 0}$$

$$\Rightarrow \frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} + \left(-\frac{u_r}{r^2}\right) = 0$$

$$\Rightarrow \frac{d}{dr}\left[\frac{du_r}{dr} + \frac{u_r}{r}\right] = 0$$

solution to this differential equation is

$$u_r = Ar + \frac{B}{r}$$

A and B are constants of Integration

$$\frac{du_r}{dr} = A - \frac{B}{r^2} \quad \text{and} \quad \frac{u_r}{r} = A + \frac{B}{r^2}$$

Hence (11) and (12)

$$\Rightarrow \sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\left(A - \frac{B}{r^2}\right) + \nu\left(A + \frac{B}{r^2}\right)\right]$$

$$= \frac{E}{(1-2\nu)(1+\nu)}\left[A - \frac{B}{r^2} - \cancel{\nu A} + \nu\frac{B}{r^2} + \cancel{\nu A} + \nu\frac{B}{r^2}\right]$$

$$\boxed{\sigma_r = \frac{E}{(1-2\nu)(1+\nu)}\left[A - (1-2\nu)\frac{B}{r^2}\right]} \rightarrow (13)$$

and

$$\boxed{\sigma_\theta = \frac{E}{(1-2\nu)(1+\nu)}\left[A + (1-2\nu)\frac{B}{r^2}\right]} \rightarrow (14)$$

Boundary conditions are @ $r = a$; $\sigma_r = -P_a$

and @ $r = b$; $\sigma_r = -P_b$.

$$\therefore \; -P_a = \frac{E}{(1-2\nu)(1+\nu)}\left[A - (1-2\nu)\frac{B}{a^2}\right] \rightarrow (15)$$

$$-P_b = \frac{E}{(1-2\nu)(1+\nu)}\left[A - (1-2\nu)\frac{B}{b^2}\right] \rightarrow (16)$$

Solve (15) and (16) to find values of A and B

$$A = \frac{(1-2\nu)(1+\nu)}{E}\left[\frac{P_b b^2 - P_a a^2}{a^2-b^2}\right]$$

$$B = \frac{1+\nu}{E}\left[\frac{P_b - P_a}{(a^2-b^2)} a^2 b^2\right]$$

Substituting these in eqns of σ_r and σ_θ

$$\Rightarrow \boxed{\sigma_r = \frac{P_a a^2 - P_b b^2}{b^2-a^2} - \frac{(P_a - P_b)}{b^2-a^2}\frac{a^2 b^2}{r^2}}$$

$$\boxed{\sigma_\theta = \frac{P_a a^2 - P_b b^2}{b^2-a^2} + \frac{P_a - P_b}{b^2-a^2}\frac{a^2 b^2}{r^2}}$$

and $\sigma_z = \nu(\sigma_r + \sigma_\theta)$

$$\boxed{\sigma_z = 2\nu \frac{P_a a^2 - P_b b^2}{b^2-a^2}}$$

Similar equations for σ_r and σ_θ will be obtained for plane stress cases.

Special Cases

Case 1: Cylinder subjected to internal pressure alone

ie $P_a \neq 0$ but $P_b = 0$

In this case

$$\sigma_r = \frac{P_a a^2}{b^2-a^2} - \frac{P_a a^2 b^2}{(b^2-a^2) r^2}$$

$$\boxed{\sigma_r = \frac{P_a a^2}{b^2-a^2}\left[1 - \frac{b^2}{r^2}\right]}$$

and

$$\sigma_\theta = \frac{P_a a^2}{b^2-a^2} + \frac{P_a a^2 b^2}{(b^2-a^2) r^2}$$

$$\boxed{\sigma_\theta = \frac{P_a a^2}{b^2-a^2}\left[1 + \frac{b^2}{r^2}\right]}$$

σ_θ is always tensile and σ_r is compressive

σ_θ is max @ $r = a$ ie @ the inner surface of the cylinder.

$$\therefore \sigma_{\theta max} = \frac{P_a a^2}{b^2-a^2}\left[1 + \frac{b^2}{a^2}\right]$$

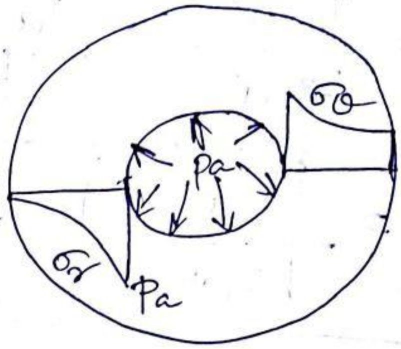
$$\boxed{\sigma_{\theta max} = \frac{P_a (a^2+b^2)}{(b^2-a^2)}}$$

Shear stress $\tau_{r\theta} = \dfrac{\sigma_\theta - \sigma_r}{2}$

$$= \frac{1}{2}\,\frac{P_a a^2}{b^2-a^2}\left[1+\frac{b^2}{r^2} - 1 + \frac{b^2}{r^2}\right]$$

$$= \frac{1}{\cancel{2}}\,\frac{P_a a^2}{(b^2-a^2)}\left[\frac{\cancel{2}b^2}{r^2}\right]$$

$$= \frac{P_a a^2 b^2}{(b^2-a^2)\, r^2}$$



$\tau_{r\theta} \propto \dfrac{1}{r^2}$

$\tau_{r\theta}$ is max @ $r = a$

$$\Rightarrow \tau_{r\theta\,max} = \frac{P_a \cancel{a^2} b^2}{(b^2-a^2)\,\cancel{a^2}}$$

$$\boxed{\tau_{r\theta\,max} = \frac{P_a b^2}{(b^2-a^2)}}$$

Case 2: When the cylinder is subjected to external pressure alone.

ie $P_a = 0$ and $P_b \neq 0$

$$\sigma_r = \frac{-P_b b^2}{(b^2-a^2)} + \frac{P_b a^2 b^2}{(b^2-a^2)\, r^2}$$

$$= \frac{P_b b^2}{(b^2-a^2)}\left[\frac{a^2}{r^2} - 1\right] = \frac{-P_b b^2}{(b^2-a^2)}\left[1-\frac{a^2}{r^2}\right]$$

$$\sigma_\theta = \frac{P_a a^2 - P_b b^2}{(b^2-a^2)} + \frac{P_a - P_b}{(b^2-a^2)} \frac{a^2 b^2}{r^2}$$

@ $P_b \neq 0$ and $P_a = 0$

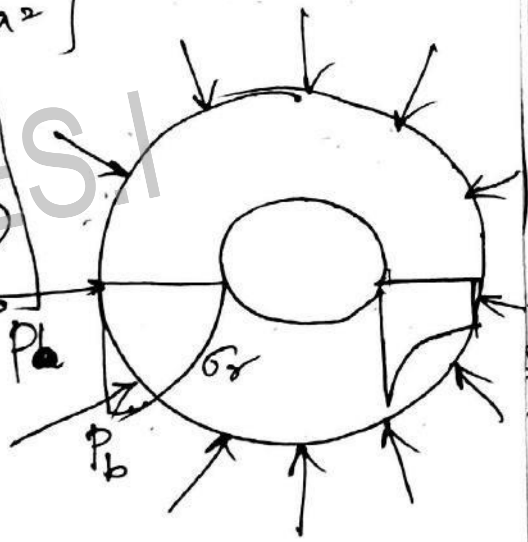
$$\Rightarrow \sigma_\theta = \frac{-P_b b^2}{(b^2-a^2)} - \frac{P_b a^2 b^2}{(b^2-a^2) r^2}$$

$$= \frac{-P_b b^2}{(b^2-a^2)} \left[1 + \frac{a^2}{r^2} \right]$$

σ_θ is max @ $r = a$

$$\sigma_{\theta max} = \frac{-P_b b^2}{(b^2-a^2)} \left[1 + \frac{a^2}{a^2} \right]$$

$$\boxed{\sigma_{\theta max} = \frac{-2 P_b b^2}{(b^2-a^2)}}$$



$$\tau_{r\theta} = \left(\frac{\sigma_\theta - \sigma_r}{2} \right)$$

$$= \frac{-P_b b^2}{(b^2-a^2)} \left[1 - \frac{a^2}{r^2} + 1 + \frac{a^2}{r^2} \right]$$

$$\boxed{\tau_{r\theta max} = \frac{-2 P_b b^2}{(b^2-a^2)}}$$

Qn.1) A thick cylinder of internal diameter 160mm is subjected to an internal pressure of $40 N/mm^2$. If the allowable stress in the material is $120 N/mm^2$ find the thickness required

$d_a = 160mm$

$r_a = 80mm = a$

$P_a = 40 N/mm^2$

Allowable stress, $\sigma_{\theta max} = 120 N/mm^2$

$\sigma_{r max} = P_a$

$$\sigma_{\theta max} = \frac{P_a}{b^2 - a^2}(a^2 + b^2)$$

$$120 = \frac{40(80^2 + b^2)}{b^2 - 80^2}$$

$\Rightarrow b = 113.14 mm$

Thickness of cylinder $= b - a$

$= 113.14 - 80$

$= \underline{\underline{33.14 mm}}$

Qn.2) A thick walled tube with an internal radius of 12cm is subjected to internal pressure of 200 MPa. Given $E = 2.1 \times 10^5$ and $\nu = 0.3$. Determine the optimum values of external radius if the max shear stress is limited to 350 MPa. Also determine the change in internal radius due to pressure?

Qn.3. and 40 interna determ

Qn.4) outer of 12N the cyl

N

Disp

U

Note that max shear stress $\tau_{max} = \frac{P_a b^2}{b^2 - a^2}$

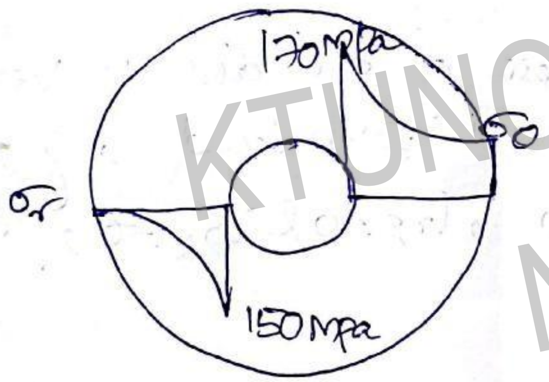
$\Rightarrow b = 183.9mm$

Displacement at the inner radius

$$u_a = \frac{a P_a}{E}\left[\frac{a^2+b^2}{b^2-a^2} + \nu\right]$$

$$= \underline{\underline{0.32mm}}$$

Qn.3. An alloy steel cylinder has 100mm internal diameter and 400mm outside diameter. If it is subjected to an internal pressure of 150 Mpa, (outside pressure = 0) determine the radial and tangential stress distribution



Qn.4) A thick cylinder of inner radius 10cm and outer radius 15cm is subjected to an internal pressure of $12N/mm^2$. Determine the radial and hoop stress in the cylinder at the inner and outer surfaces.

Radial stress, $\sigma_r = -12N/mm^2$ (@ $r=a$)

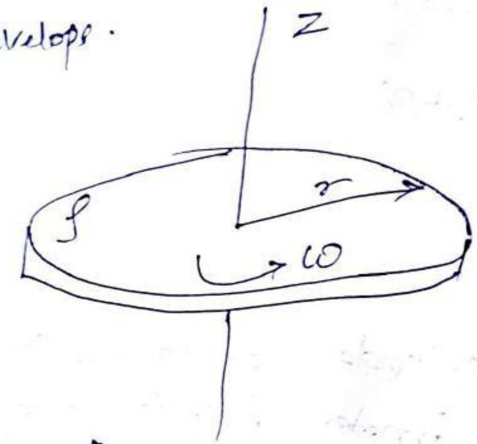
@ $r=b$, $\sigma_r = 0$

Hoop stress, $\sigma_\theta = 31.2N/mm^2$ (@ $r=a$)

$\sigma_\theta = 19.2N/mm^2$ (@ $r=b$)

Rotating Discs

When a disc is rotated, loading happens and a stress develops.



Assuming that, the density of the material is 'ρ', its angular velocity is 'ω' and radius 'r', body force will be equal to $\rho\omega^2 r$

Assume that the thickness of the disc is very small. If it is so, it cannot withstand any stress in z direction ie $\sigma_z = 0$, $\tau_{rz} = 0$ & $\tau_{\theta z} = 0$, it belongs a plane stress problem.

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The eqbm equation reduces to the following form;

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \gamma_r = 0 \quad \text{—}$$

γ_r = body force component = $\rho\omega^2 r$ $\quad kg/m^3 \times 1/s^2 \times m$

$\left(\because \frac{m\omega^2 r}{V} = \rho\omega^2 r \right.$ is the centrifugal force acting as a body force per unit volume)

$$r\frac{d}{dr}\sigma_r + \sigma_r - \sigma_\theta + \rho\omega^2 r^2 = 0 \rightarrow (1)$$

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0$$

We have $\varepsilon_r = \frac{du_r}{dr}$ and $\varepsilon_\theta = \frac{u_r}{r}$ $(\because u_\theta = 0)$

$$u_r = r\varepsilon_\theta$$

$$\Rightarrow \varepsilon_r = \frac{d}{dr}(r\varepsilon_\theta) = r$$

from Hooke's law, we have

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu\sigma_r]$$

Now; $\varepsilon_r = \frac{d}{d\theta}(r\varepsilon_\theta)$

$$= \frac{d}{dr}\left[\frac{r}{E}(\sigma_\theta - \nu\sigma_r)\right]$$

$$= \frac{1}{E}\left[\frac{d}{dr}\left(r\sigma_\theta - \nu r\sigma_r\right)\right] \rightarrow ($$

$$\cancel{\frac{1}{E}}$$

from eq: (1)

$$r\frac{d}{dr}\sigma_r + \sigma_r - \sigma_\theta + \rho\omega^2 r^2 = 0$$

$$\text{or } \frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0$$

$$\Rightarrow \boxed{\sigma_\theta = \frac{d}{dr}(r\sigma_r) + \rho\omega^2 r^2}$$

$\frac{d}{dr^2}$ $\quad = \quad -\nu \rho \omega^2 r^2$

eq: (2)

$$\Rightarrow \varepsilon_r = \frac{1}{E}\left[\frac{d}{dr}\left\{ r \cdot \left(\frac{d}{dr}(r\sigma_r) + \rho\omega^2 r^2\right) - \nu r \sigma_r \right\}\right]$$

$$= \frac{1}{E}\left\{ r \cdot \frac{d^2(r\sigma_r)}{dr^2} + \frac{d}{dr} r\sigma_r + 3\rho\omega^2 r^2 - \nu \frac{d}{dr}(r\sigma_r) \right\}$$

We have $\varepsilon_r = \frac{1}{E}(\sigma_r - \nu \sigma_\theta)$

$$= \frac{1}{E}\left(\sigma_r - \nu \left\{\frac{d}{dr}(r\sigma_r) + \rho\omega^2 r^2\right\}\right)$$

$$= \frac{1}{E}\left[\sigma_r - \nu \frac{d}{dr}(r\sigma_r) - \nu\rho\omega^2 r^2\right]$$

Comparing the above equations

$$r\frac{d^2}{dr^2}(r\sigma_r) + \frac{d}{dr}(r\sigma_r) + 3\rho\omega^2 r^2 - \sigma_r + \nu\rho\omega^2 r^2 = 0$$

Multiplying by r and rearranging

$$r^2 \frac{d^2}{dr^2}(r\sigma_r) + r\frac{d}{dr}(r\sigma_r) - r\sigma_r + (3+\nu)\rho\omega^2 r^3 = 0$$

Assume that $r\sigma_r = \psi$

The above equation reduces to

$$r^2 \frac{d^2\psi}{dr^2} + r\frac{d\psi}{dr} - \psi + (3+\nu)\rho\omega^2 r^3 = 0$$

The solution to the above differential equation is

$$\psi = Cr + \frac{D}{r} - \frac{(3+\nu)}{8}\rho\omega^2 r^3$$

ie $r\sigma_r = Cr + \frac{D}{r} - \left(\frac{3+\nu}{8}\right)\rho\omega^2 r^3$

$$\Rightarrow \boxed{\sigma_r = C + \frac{D}{r^2} - \frac{(3+\nu)}{8}\rho\omega^2 r^2}$$

Hence $\sigma_\theta = \frac{d}{dr}\left[r\sigma_r\right] + \rho\omega^2 r^2$

$$= \frac{d}{dr}\left[Cr + \frac{D}{r} - \left(\frac{3+\nu}{8}\right)\rho\omega^2 r^3\right] + \rho\omega^2 r^2$$

$$= C - \frac{D}{r^2} - \left(\frac{3+\nu}{8}\right)\rho\omega^2 \cdot 3r^2 + \rho\omega^2 r^2$$

$$= C - \frac{D}{r^2} - \frac{3}{8}\times 3\rho\omega^2 r^2 - \frac{3\nu}{8}\rho\omega^2 r^2 + \rho\omega^2 r^2$$

$$= C - \frac{D}{r^2} - \frac{9}{8}\rho\omega^2 r^2 + \rho\omega^2 r^2 - \frac{3\nu}{8}\rho\omega^2 r^2$$

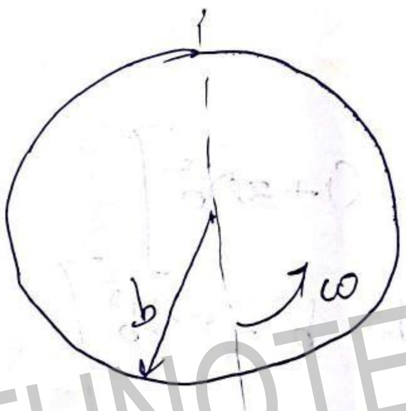
$$\Rightarrow \sigma_\theta = C - \frac{D}{r^2} - \frac{(9-8)}{8}\rho\omega^2 r^2 - \frac{3\nu}{8}\rho\omega^2 r^2$$

$$= C - \frac{D}{r^2} - \frac{\rho\omega^2 r^2}{8} - \frac{3\nu}{8}\rho\omega^2 r^2$$

$$\boxed{\sigma_\theta = C - \frac{D}{r^2} - \frac{(1+3\nu)}{8}\rho\omega^2 r^2}$$

Case 1: Solid disc

Consider a solid disc with radius 'b' and there are no external forces acting on it.



@ $r = b$, $\sigma_r = 0$

$$\Rightarrow \sigma_r = 0 = C + \frac{D}{b^2} - \left(\frac{3+\nu}{8}\right)\rho\omega^2 b^2$$

@ $r = 0$; $\sigma_r \neq \infty$; $\Rightarrow D = 0$

$$\Rightarrow \quad 0 = C - \left(\frac{3+\nu}{8}\right)\rho\omega^2 b^2$$

$$\Rightarrow C = \left(\frac{3+\nu}{8}\right)\rho\omega^2 b^2$$

$$\Rightarrow \sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho\omega^2 b^2 - \left(\frac{3+\nu}{8}\right)\rho\omega^2 r^2$$

$$\boxed{\sigma_r = \left(\frac{3+\nu}{8}\right)\rho\omega^2\left[b^2 - r^2\right]}$$

$$\sigma_\theta = C - \frac{(1+3\nu)}{8} \rho \omega^2 r^2$$

$$= \left(\frac{3+\nu}{8}\right) \rho \omega^2 b^2 - \left(\frac{1+3\nu}{8}\right) \rho \omega^2 r^2$$

$$\boxed{\sigma_\theta = \frac{\rho \omega^2}{8} \left[(3+\nu) b^2 - (1+3\nu) r^2\right]}$$

@ $r=0$; σ_r max and σ_θ max will be obtained

$$\sigma_{r\,max} = \left(\frac{3+\nu}{8}\right) \rho \omega^2 b^2$$

$$\sigma_{\theta\,max} = \left(\frac{3+\nu}{8}\right) \rho \omega^2 b^2$$

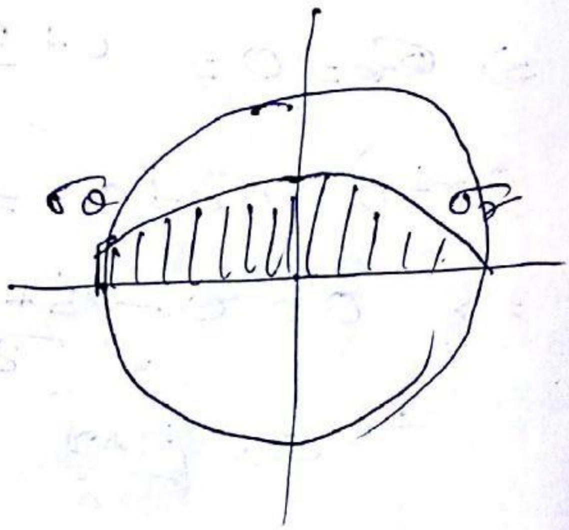
@ $r=b$;

$$\sigma_\theta = \frac{\rho \omega^2}{8} \left[(3+\nu) b^2 - (1+3\nu) b^2\right]$$

$$= \frac{\rho \omega^2}{8} \left[3b^2 + \nu b^2 - b^2 - 3\nu b^2\right]$$

$$= \frac{\rho \omega^2}{8} \left[2b^2 - 2\nu b^2\right]$$

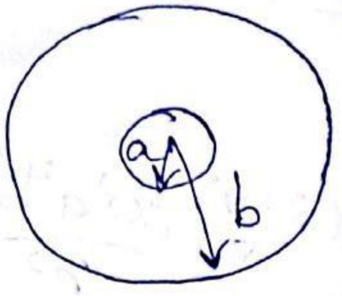
$$\boxed{\sigma_\theta = \frac{\rho \omega^2 b^2}{4} \left[1-\nu\right]}$$



@ $r=b$

$$\sigma_r = \left(\frac{3+\nu}{8}\right) \times 0 \quad = 0$$

Case 2 : Disc with a central hole



The inner and outer surface of the ~~cylinder~~ disc are not subjected to any kind of loading. ie there is loading only due to rotation of the disc

@ $r = a$; $\sigma_r = 0$ and @ $r = b$; $\sigma_r = 0$

$$\sigma_r = C + \frac{D}{r^2} - \frac{(3+\nu)}{8} \rho \omega^2 r^2$$

$$\therefore 0 = C + \frac{D}{a^2} - \left(\frac{3+\nu}{8}\right) \rho \omega^2 a^2 \longrightarrow (1)$$

and $$0 = C + \frac{D}{b^2} - \left(\frac{3+\nu}{8}\right) \rho \omega^2 b^2 \longrightarrow (2)$$

from (1) and (2)

(1) − (2)

$$\Rightarrow \frac{D}{a^2} - \frac{D}{b^2} + \left(\frac{3+\nu}{8}\right) \rho \omega^2 b^2 - \left(\frac{3+\nu}{8}\right) \rho \omega^2 a^2 = 0$$

$$\Rightarrow D\left[\frac{b^2 - a^2}{a^2 b^2}\right] + \frac{3+\nu}{8} \rho \omega^2 \left[b^2 - a^2\right] = 0$$

$$\Rightarrow \boxed{D = -\frac{(3+\nu)}{8} \rho \omega^2 a^2 b^2}$$

$$\therefore \text{from (1)} \quad C = \left(\frac{3+\nu}{8}\right) \rho \omega^2 a^2 - \frac{D}{a^2}$$

$$= \left(\frac{3+\nu}{8}\right) \rho \omega^2 a^2 + \left(\frac{3+\nu}{8}\right) \rho \omega^2 b^2$$

$$= \frac{3+\nu}{8} \rho \omega^2 (a^2 + b^2)$$

$$\boxed{\sigma_r = \left(\frac{3+\nu}{8}\right)\rho\omega^2\left[b^2+a^2-\frac{a^2b^2}{r^2}-r^2\right]}$$

$$\sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho\omega^2(b^2+a^2)+\left(\frac{3+\nu}{8}\right)\frac{\rho\omega^2a^2b^2}{r^2} - \left(\frac{1+3\nu}{8}\right)\rho\omega^2r^2$$

$\frac{1}{r}$

r^{-1-1}

$\frac{1}{r^2}$

$$\boxed{\sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho\omega^2\left[b^2+a^2+\frac{a^2b^2}{r^2}-\left(\frac{1+3\nu}{3+\nu}\right)r^2\right]}$$

To find max value of σ_r, apply the condition

$$\frac{d}{dr}(\sigma_r)=0$$

$$\Rightarrow \left(\frac{3+\nu}{8}\right)\rho\omega^2\left[-a^2b^2\left(\frac{-2}{r^3}\right)-2r\right]=0$$

$$\Rightarrow \frac{2a^2b^2}{r^3}=2r$$

$$\Rightarrow \cancel{2}a^2b^2=\cancel{2}r^4$$

$$\Rightarrow r=\sqrt{ab}$$

Hence $\sigma_{r max}$ occurs @ $r = \sqrt{ab}$

Hence $\sigma_{r max} = \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left[b^2 + a^2 - \frac{a^2 b^2}{ab} - ab \right]$

$= \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left[b^2 + a^2 - 2ab \right]$

$$\boxed{\sigma_{max} = \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left[a - b \right]^2}$$

$\frac{d}{dr}(\sigma_\theta) = 0$

$\Rightarrow \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left[-\frac{2a^2 b^2}{r^3} - \left(\frac{1+3\nu}{3+\nu}\right) \times 2r \right] = 0$

$\Rightarrow \frac{2a^2 b^2}{r^3} = -\left[\frac{1+3\nu}{3+\nu}\right] \times 2r$

$\sigma_{\theta max}$ occurs @ $r = a$

$\Rightarrow \sigma_{\theta max} = \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left[b^2 + a^2 + \frac{a^2 b^2}{a^2} - \left(\frac{1+3\nu}{3+\nu}\right) a^2 \right]$

$= \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left[2b^2 + \frac{a^2(3+\nu) - (1+3\nu) a^2}{(3+\nu)} \right]$

$= \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left[2b^2 + \frac{3a^2 + \nu a^2 - a^2 - 3\nu a^2}{(3+\nu)} \right]$

$= \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left[2b^2 + \frac{2a^2(1-\nu)}{(3+\nu)} \right]$

$= \left(\frac{3+\nu}{4}\right) \rho \omega^2 \left[b^2 + \frac{a^2(1-\nu)}{(3+\nu)} \right]$

$= \left(\frac{3+\nu}{4}\right) \rho \omega^2 \left[1 + \frac{a^2}{b^2} \left(\frac{1-\nu}{3+\nu}\right) \right]$

24/3/2017
Friday

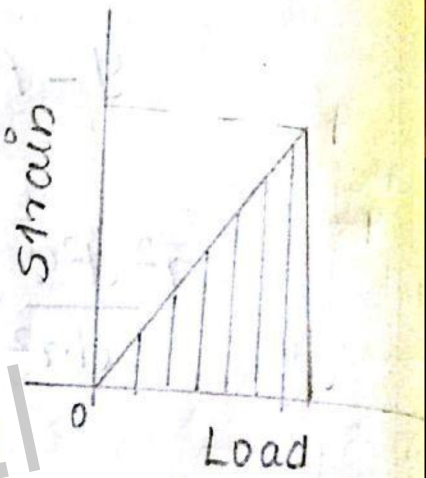
MODULE · IV

Energy Method.

Strain energy (U)

Work done $= \frac{0+P}{2} \cdot \delta$ (force x displacement)

It is the capacity to work.



$$= \frac{1}{2} \frac{P}{A} \cdot \frac{\delta}{l}$$

$$= \frac{1}{2} \sigma \, \varepsilon$$

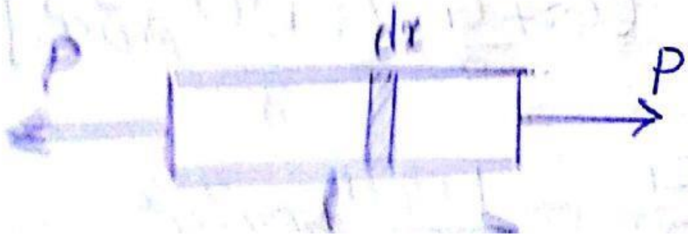
$$= \frac{\text{strain energy}}{\text{unit volume}}$$

Strain energy / unit volume is given by

$$= \frac{1}{2} \sigma \, \varepsilon$$

$$= \frac{1}{2} \sigma \cdot \frac{\sigma}{E} = \frac{\sigma^2}{2E}$$

Strain energy due to tensile force



Total strain energy.

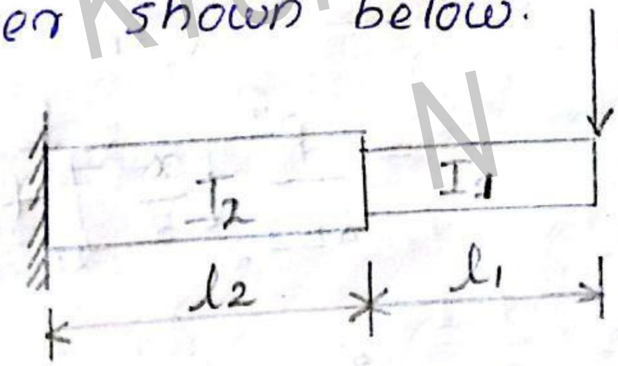
$$= \iiint \frac{\sigma^2}{2E} \, dv \qquad dv = dx\,dy\,dz$$

$$= \iiint \frac{\sigma^2}{2E} \times (A \times dx) \qquad dv = Area \times dx$$

$$U = \frac{1}{2E} \int_0^l \left(\frac{P}{A}\right)^2 \cdot A\,dx$$

$$\boxed{U = \int \frac{P^2\,dx}{2EA}}$$

- Determine the strain energy stored in a cantilever shown below.



(here loading is bending)

strain energy stored (Total) $\quad U = \int \frac{M^2\,dx}{2EI}$

consider a section x distance away from free end. Then bending moment at the section $= P.x$ (load $\times$ distance)

$$U = \int \frac{M^2 dx}{2EI}$$

U at the section $= \int_0^{l_1} \frac{(P.x)^2 dx}{2EI_1}$

next section $= \int_{l_1}^{l_2} \frac{(Px)^2 dx}{2EI_2}$

Total strain energy $= \int_{l_1}^{l_2} \frac{(Px)^2 dx}{2EI_2} + \int_0^{l_1} \frac{(Px)^2 dx}{2EI_1}$

$$= \frac{P^2}{2EI_2} \int_{l_1}^{l_2} x^2 dx + \frac{P^2}{2EI_1} \int_0^{l_1} x^2 dx$$

$$= \frac{P^2}{2EI_2} \left[\frac{x^3}{3}\right]_{l_1}^{l_2} + \frac{P^2}{2EI_1} \left[\frac{x^3}{3}\right]_0^{l_1}$$

$$= \frac{P^2}{2EI_2} \left[\frac{l_2^3}{3} - \frac{l_1^3}{3}\right] + \frac{P^2}{2EI_1} \left[\frac{l_1^3}{3}\right]$$

$$= \frac{P^2}{6EI_2} \left[l_2^3 - l_1^3\right] + \frac{P^2}{6EI_1} \left[l_1^3\right]$$

$$= \frac{P^2}{6E}\left[\frac{l_2^3 - l_1^3}{I_2}\right] + \frac{P^2}{6E}\left[\frac{l_1^3}{I_1}\right]$$

$$= \frac{P^2}{6E}\left[\frac{l_2^3 - l_1^3}{I_2} + \frac{l_1^3}{I_1}\right]$$

δ at free end $= \frac{\partial U}{\partial F_i}$

$$= \frac{\partial U}{\partial P}$$

$$= \frac{\partial}{\partial P}\left[\frac{P^2(l_2^3 - l_1^3)}{6EI_2} + \frac{P^2 l_1^3}{6EI_1}\right]$$

$$= \frac{2P(l_2^3 - l_1^3)}{6EI_2} + \frac{2P \cdot l_1^3}{6EI_1}$$

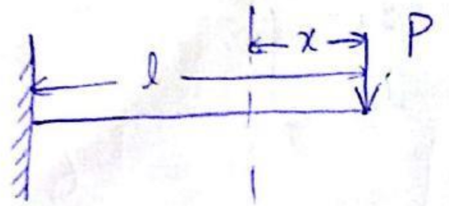
$$= \frac{P(l_2^3 - l_1^3)}{3EI_2} + \frac{P l_1^3}{3EI_1}$$

28/8/17
Tuesday

Strain energy stored in a element when it is subjected to shearing load

$$U = \int \frac{F^2 dx}{2AG}$$

Determine the deflection at the free end of the cantilever, when a point load (P) is acting at free end. length of cantilever beam, L



Total strain energy $U = \int \frac{M x^2 \, dx}{2EI}$

$M_x = Px$

$$U = \int_0^l \frac{(Px)^2 dx}{2EI}$$

$$= \frac{P^2}{2E} \left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{P^2 l^3}{6EI}$$

Tensile load $U = \int \frac{P^2 dx}{2AE}$

shear force $U = \int \frac{F^2 dx}{2AG}$

Bending $U = \int \frac{M^2 dx}{2EI}$

Torque $U = \frac{T^2 dx}{2}$

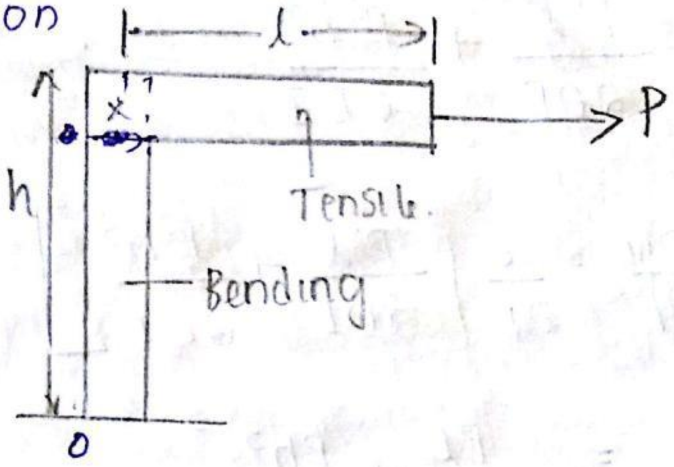
$$\delta i = \frac{\partial u}{\partial P}$$

$$= \frac{\partial}{\partial p}\left(\frac{P^2 l^3}{6EI}\right)$$

$$\delta i = \frac{2Pl^3}{6EI}$$

$$= \frac{Pl^3}{3EI}$$

Determine the strain energy stored in the following system and find out the corresponding deflection



Findout strain energy

$$U_{tensile} = \int_0^l \frac{P^2 dx}{2AE}$$

$$= \frac{P^2}{2AE}[x]_0^l$$

$$= \frac{P^2 l}{2AE}$$

$$U_{Bending} = \int \frac{M^2 dh}{2EI}$$

$$= \int_0^h \frac{(Ph)^2 dh}{2EI}$$

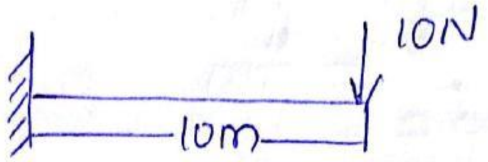
$$= \frac{P^2}{2EI}\left[\frac{h^3}{3}\right]_0^h = \frac{P^2 h^3}{6EI}$$

Total $U = U_{tensile} + U_{Bending}$

$$= \frac{P^2 l}{2AE} + \frac{P^2 h^3}{6EI}$$

$$\delta = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left[\frac{P^2 l}{2AE} + \frac{P^2 h^3}{6EI}\right]$$

$$= \frac{Pl}{AE} + \frac{Ph^3}{3EI}$$



Determine the deflection at the free end. (using energy method)

A strain tensor $\varepsilon_{ij} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 3/2 \\ -1 & 3/2 & 0 \end{bmatrix} \times 10^{-3}$

$$E = 207 \times 10^6 \text{ kPa}$$

$$G = 80 \times 10^6 \text{ kPa}$$

Determine the value of strain energy density

strain energy density = strain energy / unit volume

$$= \frac{1}{2}\left[\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right]$$

$\cancel{\sigma_x = 2G E_x +}$

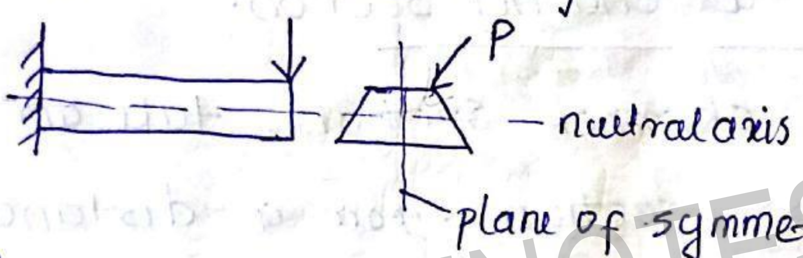
$$E = 2G(1+\mu)$$

Find $\mu = ?$

$$\lambda = \frac{E\mu}{(1+\mu)(1-2\mu)}$$

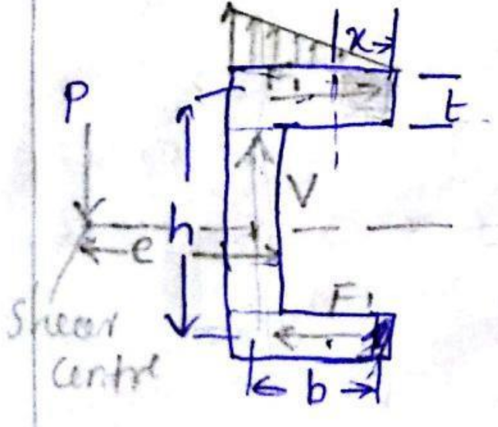
4/4/17
Tuesday

- unsymmetrical bending



In unsymmetrical bending,
The transverse load or bending moment acts not in the plane of symmetry, the neutral axis for unsymmetrical bending is not $\perp r$ to the direction of transverse load.

- shear centre



I. $\tau = \frac{VQ}{It}$

II. Apply bending moment

III. Take moment about centroid.

When a beam has a unsymmetrical cross section, if the load passes through the centre of the beam of the cross section, then there is a twisting in addition to the bending of the beam.

In order to avoid twisting of the beam and to have pure bending load has to be applied through some appropriate point. This point is called shear centre.

Shear centre of a channel section.

consider a channel section, take an arbitrary cross section from a distance x at top flange.

$$\tau = \frac{V \times Q}{It}$$

Q = area moment

V = Shear force.

$$= \frac{V \times (t \cdot x) h/2}{It}$$

$$= \frac{Vh}{2I} x$$

when $x = 0_{min}$ $\tau = 0$

$x = b_{max}$ $\tau = \frac{Vh}{2I} b$

Average ~~fo~~ shear force F_1 $= \left[\dfrac{0 + \frac{Vh}{2I} b}{2} \right] \times bt$

Average shear stress

area of cross section.

$$F_1 = \frac{Vhb^2t}{4I}$$

Let P be the applied load at a distance e from the web centere line. To maintain this applied force in equilibrium. An equal and opposite shearing force must be developed in the web.

In order to cause not twisting of channel the couple produced by the applied load

$$P \times e = F_1 \times \frac{h}{2} + F_1 \times \frac{h}{2}$$

$$= F_1 \times h$$

$$e = \frac{F_1 \times h}{P}$$

$$= \frac{Vhb^2th}{4IP} = \frac{Vh^2b^2t}{4IP}$$

$$e = \frac{h^2b^2t}{4I}$$

$V = P$

(Shearforce = applied force to maintain equilibrium).

moment of inertia $= I_{web} + Ad^2$

$$I = \frac{th^3}{12} + 2 \cdot bt \left(\frac{h}{2}\right)^2$$

$$= \frac{th^3}{12} + \frac{bth^2}{2}$$

$$e = \frac{h^2 b^2 t}{4\left[\frac{th^3}{12} + \frac{h^2 bt}{2}\right]}$$

$$= \frac{h^2 b^2 t}{4\left[\frac{th^3 + 6h^2 bt}{12}\right]}$$

$$= \frac{3h^2 b^2 t}{th^3 + 6h^2 bt}$$

$$= \frac{3h^2 b^2 t}{th^2 (h + 6b)}$$

$$e = \frac{3b^2}{h+6b}$$

$$\tau = \frac{VQ}{It}$$

V = Shearing force developed resisting for the applied load

Q = area moment of section about neutral axis.

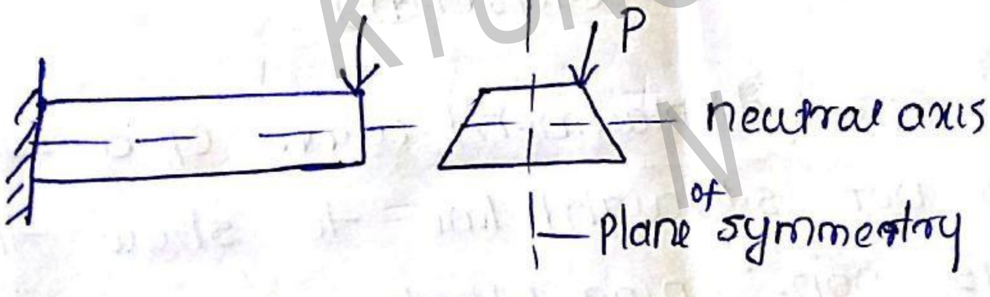
I = moment of inertia about neutral axis
t = thickness of the section.

REVISION

Define shear centre

~~In order t~~ when a beam has unsymmetrical cross section; if the load passes through the centroid of the beam of cross section, then there is a twisting in addition to the bending of the beam in order to avoid twisting of the beam and to have pure bending load load has to be applied through ~~sam~~ some appropriate point

unsymmetrical bending.



How will you solve elasticity problem ~~by~~ using strain energy method.

1. Find out total strain energy U

Then $\delta_i = \dfrac{\partial U}{\partial P}$

Determine $\sigma_r, \sigma_\theta, \tau_{r\theta}$.

Lamis problem
strain energy
ϕ

when $\phi = r^2\cos\theta$

$$\sigma_\theta = \frac{\partial \phi}{\partial r} = \frac{\partial}{\partial r}(r^2\cos\theta) = 2\cos\theta$$

2017
May.

complementary energy

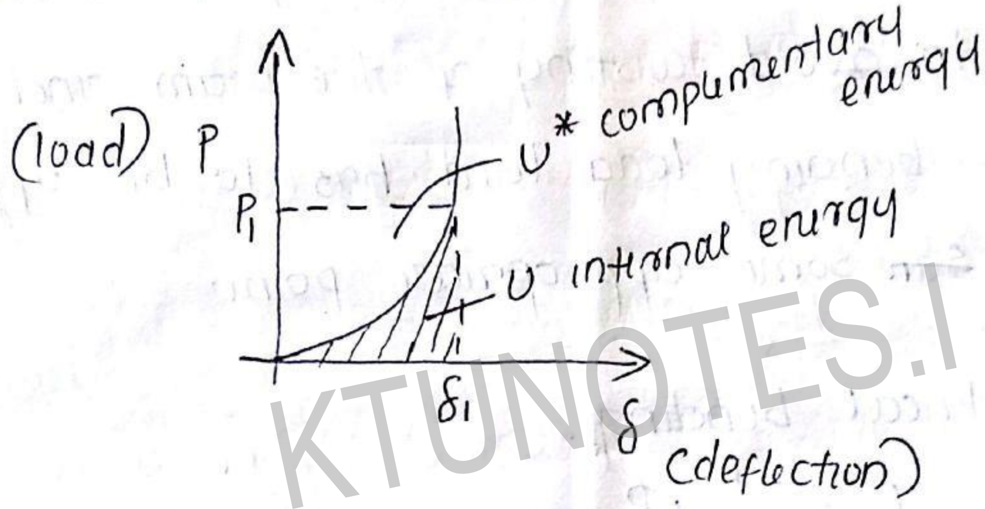
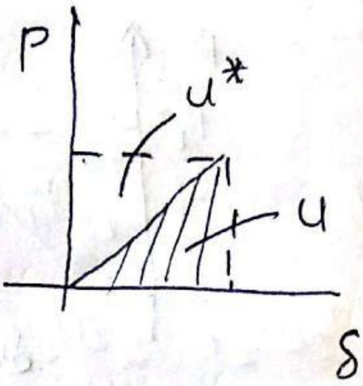


Fig shows force displacement curve of a body. The curve is not straight line to show that the body is non-linear elastic. The equilibrium displacement corresponding to force P_1 is δ_1. The area below the curve is the strain energy, which represented by U. The area between the curve and vertical axis is called the complementary energy U^*

Strain energy $U = \int^{\delta} P\,d\delta$

complementary energy $U^* = \int_0^P \delta \, dp$

Linear elastic element



For a linear elastic element load deflection curve will be as shown below.

castigliano's 1st theorem

If the strain energy of a body is expressed as a function of deflection along the direction of applied loads, the material and geometrical properties of the body, then partial derivative of strain energy w.r.t deflection at one of the point is equal to the load acting at that point

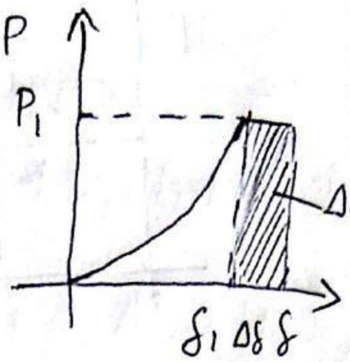
$$\boxed{\frac{\partial U}{\partial \delta_i} = P_i}$$

Proof

$$\Delta U = P_1 \Delta \delta_1 + P_2 \Delta \delta_2 + M_3 \Delta \theta_3$$

$$\frac{\Delta U}{\Delta \delta_1} = P_1$$ using principle of virtual displacement.

$$\lim_{\Delta \delta_1 \to 0} \frac{\Delta U}{\Delta \delta_1} = \frac{\partial u}{\partial \delta i} = P_1$$



$$\frac{\partial u}{\partial \delta i} = P_i$$

Castigliano's second theorem

$$\boxed{\frac{\partial U}{\partial P_i} = \delta i}$$

Using principle of virtual force

$$\cancel{\Delta U = \Delta P_1 \Delta \delta_1 + \Delta P_2 \Delta \delta_2 + \Delta M_3 \Delta \theta_3}$$

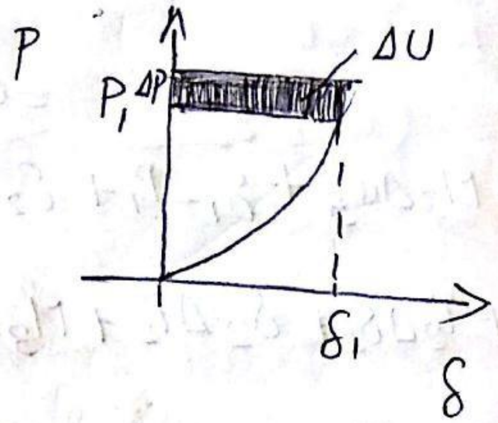
$$\Delta U = \delta_1 \Delta P_1 + \delta_2 \Delta P_2 + \theta_3 \Delta M_3$$

$$\frac{\Delta U}{\Delta P_1} = \delta_1$$

$$\lim_{\Delta P_1 \to 0} \frac{\Delta U}{\Delta P_1} = \frac{\partial U}{\partial P_1} = \delta_1$$

$$\therefore \frac{\partial U}{\partial P_i} = \delta_i$$

If the strain energy of a linear elastic body is expressed as deflections along the direction of applied load and properties of the body and its material, The partial derivative of the strain energy w.r.t one of the loads is equal to the displacement along the direction of that load.



Principle of minimum potential Energy.

The principle of ~~mo~~ minimum potential energy states that th equilibrium displacement of an elastic solid under the action of a load or system of loads is the one having the minimum net potential energy. Net potential energy is the difference between strain energy (U) and the work done by applying load (W).

$$\frac{\partial(U-W)}{\partial \delta} = \text{zero}$$

Proof of principle of minimum potential energy theory.

$$\frac{\partial}{\partial \delta}(U-W)=0$$

$$= P_1\Delta\delta_1 + P_2\Delta\delta_2 + M_3\Delta\theta_3 + \overbrace{\delta_1\Delta P_1 + \delta_2\Delta P_3 + \theta_3\Delta M_3}^{=0}$$

$$= P_1\Delta\delta_1 + \delta_1\Delta P_1 + P_2\Delta\delta_2 + \delta_2\Delta P_2 + M_3\Delta\theta_3 + \theta_3\Delta M_3$$

$$= \Delta(P_1\delta_1) + \Delta(P_2\delta_2) + \Delta(M_3\theta_3)$$

$$= \Delta(P_1\delta_1 + P_2\delta_2 + M_3\theta_3)$$

$$= \Delta W$$

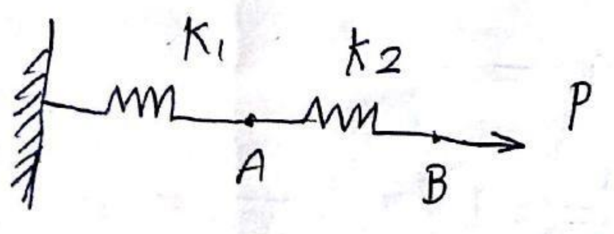
$$\Delta(U-W)=0$$

$$\frac{\Delta(U-W)}{\Delta\delta}=0$$

$$\lim_{\Delta\delta\to 0}\frac{\Delta(U-W)}{\Delta\delta} = \frac{\partial}{\partial\delta}(U-W)=0$$

$$\frac{\partial}{\partial\delta}(U-W)=0.$$

Determine the deflection at point A and B



$\delta A = ?$ $\delta B = ?$

$$\frac{\partial}{\partial \delta}(U-W) = 0$$

$$U - W = \frac{1}{2}k_1\delta_A^2 + \cancel{\frac{1}{2}k_2\delta_B(\delta_A-\delta_B)}\ \overbrace{\frac{1}{2}k_2(\delta_B-\delta_A)^2}^{U} - P\cdot\delta_B$$

$$\cancel{\frac{\partial}{\partial \delta_A}(U-W) = \frac{1}{2}k_1(\delta_A^2 -}$$

$$\cancel{\frac{\partial}{\partial \delta_A}(U-W) = \frac{1}{2}\delta_A^2(k_1-k_2) + \frac{1}{2}k_2\delta_B^2}$$

$$\frac{\partial}{\partial \delta}(U-W) = 0$$

$$U - W = \frac{1}{2}k_1\delta_A^2 + \frac{1}{2}k_2(\delta_B-\delta_A)^2 - P\delta_B$$

$$= \frac{1}{2}k_1\delta_A^2 + \frac{1}{2}k_2(\delta_B^2 - 2\delta_B\delta_A + \delta_A^2) - P\delta_B$$

$$\frac{\partial}{\partial \delta_A}(U-W) = \frac{1}{2}k_1\cdot 2\delta_A + \frac{1}{2}k_2(\delta_B^{2\,\to 0} - 2\delta_B + 2\delta_A) - 0$$

$$= k_1\cdot\delta_A + k_2(\delta_A - \delta_B) = 0 \quad \text{——— (1)}$$

$$\frac{\partial}{\partial \delta_B}(U-W) = \frac{1}{2}k_1 \times 0 + \frac{1}{2}k_2(2\delta_B - 2\delta_A) - P$$

$$= k_2(\delta_B - \delta_A) - P = 0 \quad \text{——— (2)}$$

$$k_1\cdot\delta_A + k_2(\delta_A - \delta_B) = 0$$

$$k_2(\delta_B - \delta_A) - P = 0$$

$$k_1\delta_A + k_2\delta_A - k_2\delta_B = 0$$

$$(k_1+k_2)\delta_A = k_2\,\delta_B$$

$$\delta_A = k_2 \cdot \frac{\delta_B}{(k_1+k_2)}$$

$$k_2 = \frac{P}{\delta_B-\delta_A}$$

$$① = \quad k_1\delta_A + \frac{P}{\delta_B-\delta_A}(\delta_A-\delta_B) = 0$$

$$k_1\,\delta_A - P = 0$$

$$\delta_A = \frac{P}{k_1}$$

$$(2) \rightarrow \quad \frac{P}{\cancel{\delta_B-\delta_A}}\cancel{(\delta_B-\delta_A)} + P = 0$$

$$k_2\left(\delta_B - \frac{P}{k_1}\right) = P$$

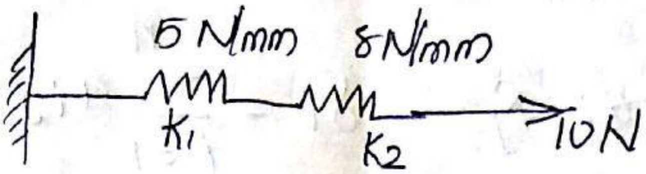
$$k_2\,\delta_B - \frac{k_2 P}{k_1} = P$$

$$k_2\,\delta_B = P + P\cdot\frac{k_2}{k_1}$$

$$= P\left(1+\frac{k_2}{k_1}\right)$$

$$\delta_B = \frac{P}{k_2}\left(1+\frac{k_2}{k_1}\right)$$

determine the deflection at point A and B



$\delta_A = ?$ $\delta_B = ?$

$$\delta_A = \frac{P}{k_1} = \frac{10}{5} = 2 \text{ N/mm}$$

$$\delta_B = \frac{P}{k_2}\left(1+\frac{k_2}{k_1}\right) = \frac{10}{8}\left(1+\frac{8}{5}\right)$$

$$= \frac{5}{4}\left(1+\frac{8}{5}\right)$$

$$= \frac{5}{4} \cdot \frac{13}{5} = \frac{13}{4} \text{ mm}$$

$$= 3.25 \text{ mm}$$

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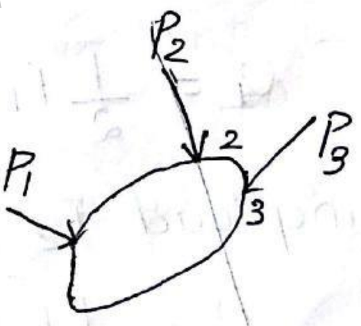
Wednesday

Reciprocal theorey [$a_{ij} = a_{ji}$]

$\sigma \propto \varepsilon$
$\sigma = \text{a const} \cdot \varepsilon$



Influence coefficient $= \delta_1 \propto P_1$ deflection produced at point 1.

$$\delta_1 = a_{11} P_1 + a_{12} P_2 + a_{13} P_3$$

$$\delta_2 = a_{21} P_1 + a_{22} P_2 + a_{23} P_3$$

$$\delta_3 = a_{31} P_1 + a_{32} P_2 + a_{33} P_3$$

$\cancel{\delta_i = a_{ij} F_1}$

$\cancel{\delta_i = a_{ij} P_j}$ $$\delta_i = \sum_{j=1}^{n} a_{ij} P_j$$

influence coefficent = partial displacement at i along the direction of P_i due to unit load

$\cancel{\delta_i = \sum}$

The influence co-efficient at point j due to

$$\delta_i = \sum_{j=1}^{n} a_{ij} P_j$$

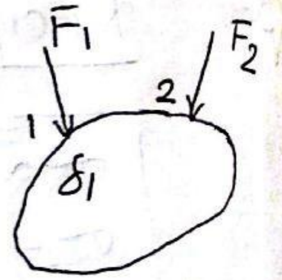
The influence co-efficcent at point j due to load acting at point i

$$U_1 = \frac{1}{2} F_1 \delta_1$$

a_{ij} — i: The point at which deflection produced; j: The point at which load is applied.

$$= \frac{1}{2} F_1 (a_{11} F_1 + a_{12} F_2 + \dots)$$

Let F_1 is acting along initially



$$U_1 = \frac{1}{2} F_1 a_{11} F_1$$

By applying load F_2

$$U_2 = \frac{1}{2}\left[F_2 (a_{22} F_2) + F_2 (a_{12} F_1)\right]$$

Total $U = U_1 + U_2$

$$= \frac{F_1}{2} a_{11} F_1 + \frac{F_2}{a_2} a_{22} F_2 + \frac{F_2}{2} a_{12} F_1$$

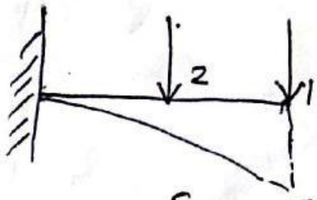
If F_2 is applied first

Total strain energy $U' = \frac{F_2}{2} a_{22} F_2 + \frac{F_1}{2} a_{11} F_1 + \frac{F_1}{2} a_{21} F_2$

But $U = U'$

$$\frac{F_1}{2} \cancel{a_{11} F_1} + \frac{F_2}{2} \cancel{a_{12} F_2} + \frac{F_2}{2} a_{12} F_1 = \frac{F_2}{2} \cancel{a_{22} F_2} + \frac{F_1}{2} \cancel{a_{11} F_1} + \frac{F_1}{2} a_{21} F_2$$

$$a_{12} = a_{21}$$

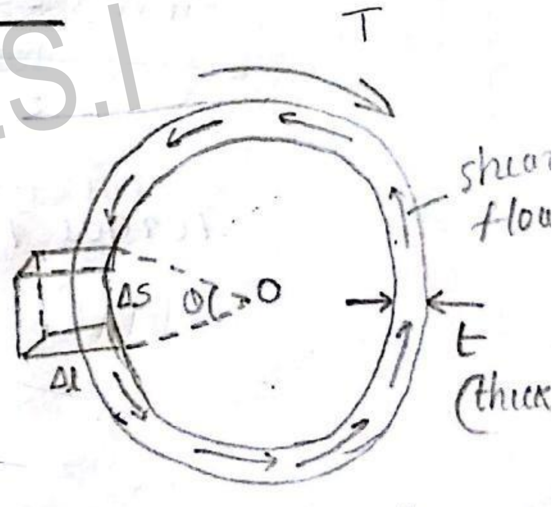


$\delta_{12} = \delta_{21}$

17
day

Torsion of thin walled tube.

Consider the equilibrium of an element of length Δl and width Δs.



Let τ_1 and τ_2 are the complementary shear stresses

Apply equilibrium equations

$$\tau_1 \times t_1 \times \Delta l = \tau_2 t_2 \Delta l$$

$$\tau_1 t_1 = \tau_2 t_2$$

$$\tau t = \text{a constant } (q)$$

constant is called shear flow.

Shear flow $(q) = \frac{\text{shear for}}{\text{unit len}}$

$= \tau \times t$ (thic

$= \frac{N}{m^2} \times m$

$= N/m$

equation is similar to the flow of an incompressible liquid in a tube of varying area of cross section. For continuity we have $A_1V_1 = A_2V_2$

$$\text{Torque} = \int r \tau t \, ds$$

$$= \tau t \int r \, ds$$

$$= \tau t \int_0^{2\pi} r \cdot r \, d\theta$$

$$= \tau t \cdot r^2 [\theta]_0^{2\pi}$$

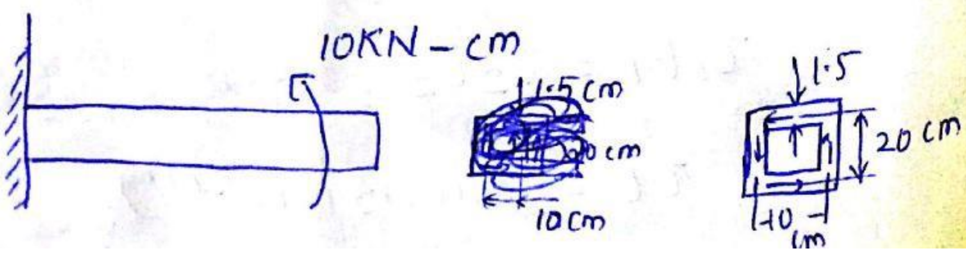
$$= 2 \tau t r^2 \pi$$

$$= 2 \tau t A$$

$$\text{Torque } T = 2 \tau t A$$

$$= \underline{\underline{2qA}}$$

Determine the shear flow, maximum shear stress reduce for a thin section cantilever beam shown in fig.



Thickness = 1.5 cm

Torque $T = 10\ KN\text{-}cm$

$$T = 2qA$$

$$A = 20 \times 10\ cm^2$$

$$q = \frac{T}{2A} = \frac{10 \times 10^3}{(20 \times 10)2}$$

$$= 25\ N/cm$$

$$q = \cancel{25 \times 10^{2}}\ 25\ N/cm$$

maximum shear stress

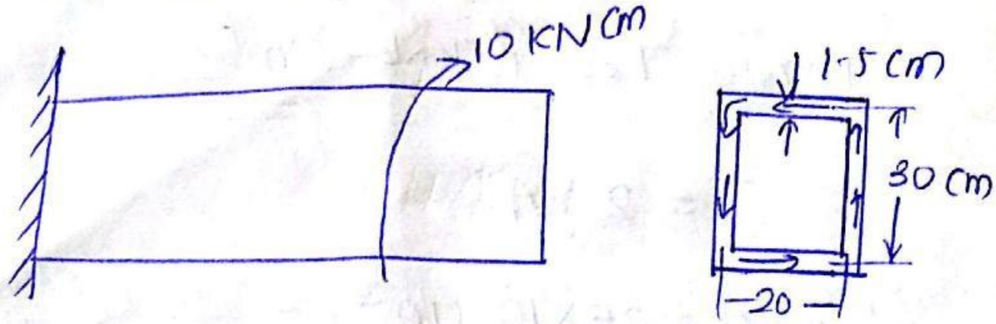
$$T = 2\tau t A$$

$$\tau_{max} = \frac{T}{2tA}$$

$$= \frac{10 \times 10^3}{2 \times 1.5 \times 20 \times 10}$$

$$= 16.66\ N/cm^2$$

(•) Determine the shear flow (q) and maximum shear stress for thin section cantilever shown in figure.



Shear flow

$$T = 2qA$$

$$q = \frac{T}{2A}$$

$$= \frac{10 \times 10^3 \text{ N cm}}{2 \times 30 \times 20}$$

$$q = 8.33 \text{ N/cm}$$

maximum shear stress

$$\tau_{max} = \frac{q}{t}$$

$$= \frac{8.33}{1.5}$$

$$= 5.55 \text{ N/cm}^2$$

Angle of twist for thin walled tubes

Shear force $= \tau t . \Delta s$

$= q \cdot \Delta s$

$\frac{\partial U}{\partial T}$

castiglianos 2nd theorem

Strain energy ΔU

$= \frac{1}{2} q \Delta s \delta$

$= \frac{1}{2} q \Delta s [\gamma \cdot \Delta l]$

$= \frac{1}{2} q \Delta s \cdot \frac{\tau}{G} \cdot \Delta l$

$\Delta U = \frac{1}{2} q \Delta s \cdot \frac{q}{tG} \Delta l$

$\Delta U = \frac{q^2 \Delta s \Delta l}{2tG}$ $\quad T = 2qA$

$\Delta U = \frac{\left(\frac{T}{2A}\right)^2 \Delta s \cdot \Delta l}{2Gt}$

Total strain energy $\Delta U = \frac{T^2 . \Delta s . \Delta l}{8A^2 G t}$

$\theta = \frac{\partial U}{\partial T}$

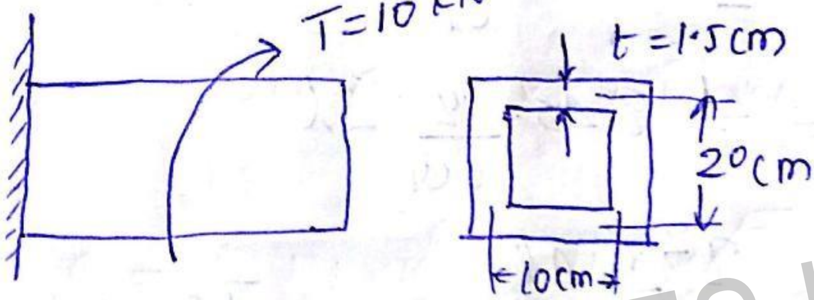
$= \frac{\partial}{\partial T} \left[\frac{T^2 \Delta s \cdot \Delta l}{8 A^2 G t} \right]$

$= \frac{2T \cdot \Delta s \cdot \Delta l}{\cancel{8A}\, 8A^2 G t}$

θ per unit length $= \dfrac{2T\Delta S}{8A^2GIt}$

$= \dfrac{TS}{4A^2Gt}$ — perimeter

Determine shear flow, maximum shear stress and angle of twist per unit length for a thin section cantilever as shown in fig.



$q = \dfrac{T}{2A} = \dfrac{10\times10^3}{2\times20\times10} = \underline{\underline{25\ N/cm}}$

$\tau_{max} = \dfrac{q}{t} = \dfrac{25}{1.5} = \underline{\underline{16.66\ N/cm^2}}$

$\theta = \dfrac{TS}{4A^2Gt}$

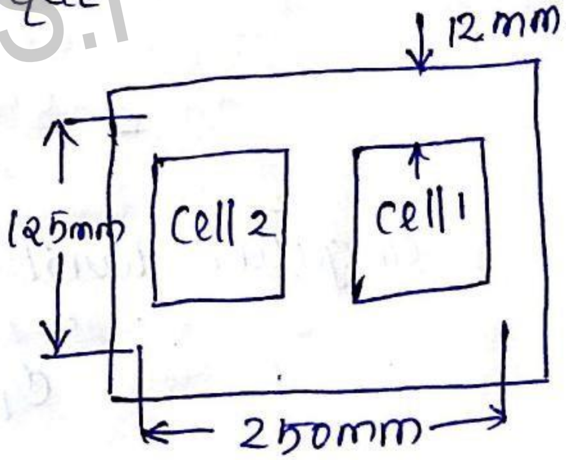
$= \dfrac{10\times10^3\times60}{4\times(20\times10)^2\times1.5\times G}$

$$\theta = \frac{q}{2AG}\oint \frac{ds}{t} \qquad T = 2\tau t A$$
$$= 2qA$$

$$= \frac{Ts}{4A^2Gt}$$

For multiple cell $T = T_1 + T_2$
$$= 2q_1A_1 + 2q_2A_2$$

A steel grid having c.s wall thickness is uniformly 12 mm the shear stress due to twisting shouldn't exceed 350,000 Pa. neglect stress concentration Determine the max. allowable torque



Given

$t = 12\,mm$

$\tau_{max} = 350{,}000\,Pa$

$T = T_1 + T_2$
$$= 2q_1A_1 + 2q_2A_2$$

$\cancel{T = 2qA}$ Both cells are symmetrical.

$$A_1 = 125 \times 125 = 15625\,mm^2$$
$$= 15625 \times 10^{-6}\,m^2$$

$\cancel{T = 4 \times [350000 \times 12\times10 ...}$

$$T = 4\left[350000 \times 12\times10^{-3} \times 125\times125\right]$$

$$= \underline{\underline{262.5}}\ N/m$$

$$\theta = \frac{TS}{4A^2Gt}$$

$$= \frac{262.5 \times \cancel{...}\times10^{-4}\left[\overset{0.250}{\cancel{.225}} + .125 + .250 + .1...\right.}{4\times(0.125\times0.125)^2 \times G \times 0.012}$$

$$\theta = \frac{\cancel{14\times10^6}}{\cancel{G}}\ \frac{\cancel{42\times10^6}^{5}}{G} = \frac{42\times10^5}{G}\ rad$$

$$=$$

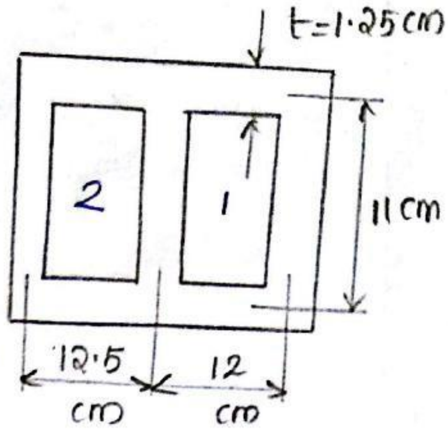
angular twist for cell 1 is given by

$$\theta_1 = \frac{1}{2GA_1}\left[q_1 \times \oint_{cell_1}\frac{ds}{t} - q_2\oint_{web}\frac{ds}{t}\right]$$

Similarly

$$\theta_2 = \frac{1}{2GA_2}\left[q_2\oint_{cell_2}\frac{ds}{t} - q_1\oint_{web}\frac{ds}{t}\right]$$

Thin walled sections with multiple closed cell



$$T = \cancel{T_1 + T_2}\ T_2 + T_1$$

$$= 2q_2A_2 + 2q_1A_1$$

$$= 2q_2(12.5\times 11) + 2q_1(12\times 11)$$

$$= 2q_2\times(12.5\times 11)\times 10^{-4} + 2q_1(12\times 11)\times 10^{-4}$$

$$\tau_{max} = 350\times 10^{6}\,Pa$$

$$T = 0.0264\,q_1 + 0.0275\,q_2 \text{ N-m}$$

$$= \cancel{(0.264\times 350\times 10^{6}\times 1.25\times 10^{-2}) + (0.0275\times 350\times 10^{6}\times 1.25\times 10^{-2})}$$

$$= \cancel{2.35\times 10^{-7}\ N/m}$$

$$\theta_1 = \frac{1}{2GA_1}\left[q_1\oint_{cell_1}\frac{ds}{t} + q_2\oint_{web}\frac{ds}{dt}\right]$$

$$\theta_1 = \frac{1}{2G\times(12\times 11)}\left[q_1\times\left[\frac{2\times 12}{1.25} + \frac{2\times 11}{1.25}\right] - q_2\left[\frac{11}{1.25}\right]\right]$$

$$\theta_2 = \frac{1}{2AG(12.5\times 11)}\left[q_2\times\left[\frac{2\times 12.5}{1.25} + \frac{2\times 11}{1.25}\right] - q_1\left[\frac{11}{1.25}\right]\right]$$

For continuity of the cross section, the angle of twist in the 2 cells are equal

when $\theta_1 = \theta_2$

$$\frac{1}{12\times 11}\left[q_1\,36.8 - q_2\,8.8\right] = \frac{1}{12.5\times 11}\left[q_2\times 37.6 - q_1\times 8.8\right]$$

$$\frac{36.8\,q_1}{12\times 11} + \frac{8.8\,q_1}{12.5\times 11} = \frac{q_2\times 37.6}{12.5\times 11} + \frac{q_2\times 8.8}{12\times 11}$$

$$\frac{5060\,q_1 + 1161.6\,q_1}{18150} = \frac{4963.2\,q_2 + 1210\,q_2}{18150}$$

$$6221.6 q_1 = 6173.2 q_2$$

$$\frac{q_1}{q_2} = \frac{6173.2}{6221.6}$$

$$\frac{q_1}{q_2} = 0.9922$$

$$q_2 = \frac{1}{0.9922} q_1$$

$$q_2 = 1.0078 q_1$$

$$T = T_1 + T_2$$

$$= 0.0264 q_1 + 0.0275 q_2$$

$$= 0.0264 q_1 + 0.0275 \times 1.0078 q_1$$

$$= 0.0264 q_1 + 0.02771 q_1$$

$$= \underline{\underline{0.05411 q_1}}$$

$$q_{max} = \tau_{max} \times t$$

$$= 350 \times 10^6 \times 1.25 \times 10^{-2}$$

$$= \underline{\underline{4.375 \times 10^6 \ N/m^2}}$$

$$q_2 > q_1$$

$$\therefore q_{max} = q_2 = 4.375 \times 10^6$$

$$q_1 = 0.9922 \times q_2$$

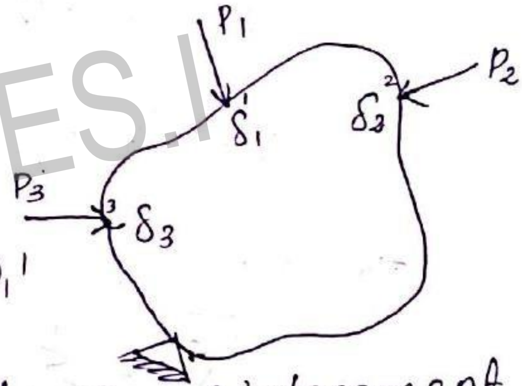
$$= 434.08 \times 10^4 \ Nlm$$

MODULE - 5

Reciprocal Theoram

* ~~Lenier~~ Linear elasticity & Hooke's law.

"A body having a strain line variation b/w load & displacement is called a lenier elastic body. By hook's law the stress at a point in a lenier elastic body is related to this strain by a set of lenier equation (constitutive relation). Hooks law can be stated in a different manner in which the deflections at any point on the body is related to the external loads



Consider a body acted upon by single force 'P_1' at the point '1'.

By hook's law the component of displacement at point '1' along the direction of force 'P_1' is directly propotional to 'P_1'. ~~ie~~ ~~$\delta_1 \propto P_1$~~

ie, $\delta_1 \propto P_1$

$$\delta_1 = a_{11} P_1$$

where a_{11} - propotionality constant which depends on the size, shape and meterial properties, and is usually known as influence coeficent of displacement at point 1, due to the load acting at point 1.

Now, suppose the body is acted upon by a 2nd load 'P_2' acting at the point 2. the load 'P_2' also brings a deformation at '1'. The partial displacement at '1' due to the load P_2 is proportional to load P_2 and is equal to $a_{12} P_2$. Influence coefficient of displacement at ① due to the load acting at ②. Now from the figure the total displacement at each points are given by

$$\delta_1 = a_{11}P_1 + a_{12}P_2 + a_{13}P_3$$

$$\delta_2 = a_{21}P_1 + a_{22}P_2 + a_{23}P_3$$

$$\delta_3 = a_{31}P_1 + a_{32}P_2 + a_{33}P_3$$

In general

$$\boxed{\delta_i = \sum_{j=1}^{n} a_{ij} P_j}$$

Reciprocal theorem states that for a linear elastic solid the influence coefficient of displacement at the point 'i' due to a load at 'j' is equal to the influence coefficient of displacement at the point 'j' due to the load acting at 'i'

ie $a_{ij} = a_{ji}$

Proof.

Here, let us imagine a situation of loading in which the body is acted upon by only the force P_1 at the point '1'.

$$\delta_1 = a_{11}P_1$$

The strain energy stored in the body due to this load P_1.

$$U_1 = \tfrac{1}{2} P_1 \delta_1$$

$$= \frac{1}{2} P_1 (a_{11} P_1).$$

$$U_1 = \frac{1}{2} P_1^2 a_{11}$$

Now under this condition let the body we applied with a force 'P_2' at the point ② which produces a deformation at point ① along the direction of P_1. Due to this deformation $a_{12}P_2$, there produces a work along the direction of P_1.

$$a_{12}P_2 \rightarrow \text{along } P_1.$$

$$W = P_1 \times a_{12} P_2$$

$$= a_{12} P_1 P_2$$

This workdone is stored as an additional energy and hence the total strain energy at '1' becomes

$$U_1 = \tfrac{1}{2} a_{11} P_1^2.$$

Also the strain energy due to load 'P₂' at point '2' is given by

$$U_2 = \frac{1}{2} a_{22} P_2^2$$

Total energy of the system,

$$U = U_1 + U_2$$

$$= \frac{1}{2} a_{11} P_1^2 + a_{12} P_1 P_2 + \frac{1}{2} a_{22} P_2^2 \quad \text{---(3)}$$

Now if we assume that 'P_2' is applied prior to P_1. Then the deformation at point 2 is

$$\delta_2 = a_{22} P_2$$

Strain energy at point 2 becomes.

$$U_2 = \frac{1}{2} P_2 \delta_2 = \frac{1}{2} a_{22} P_2^2$$

From this condition if we applying the point load 'P_1' at point ①, it produces a deformation of $a_{21} P_1$ at point ②. in the direction of P_2, There by produces a work in the direction of 'P_2', and is given by

$$W = P_2 (a_{21} P_1)$$

$$= a_{21} P_1 P_2$$

Also strain energy at point '1' becomes

$$U_1 = \frac{1}{2} P_1 \delta_1$$

$$= \frac{1}{2} a_{11} P_1^2.$$

Strain energy at point '2' is

$$U_2 = \frac{1}{2} a_{22} P_2^2 + a_{21} P_1 P_2$$

∴ The total strain energy,

$$U' = U_1 + U_2$$

$$U' = \frac{1}{2} a_{11} P_1^2 + \frac{1}{2} a_{22} P_2^2 + a_{21} P_1 P_2 \quad \text{---(4)}$$

From equ -(3) & (4) $U = U'$

Since total energy is constant.

$$\Rightarrow a_{21} = a_{12}$$

It is reciprocal theorem.

★ Maxwells Reciprocal theoram

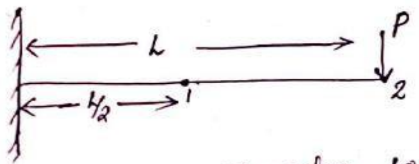
P_1, P_2, P_3 acting on a body that produces equilibrium displacement $\delta_1, \delta_2, \delta_3 \ldots$ at the point of application of load. Let $P_1', P_2', P_3' \ldots$ is an another set of forces acting at a same points and in the same direction as that of the first system of forces, and let $\delta_1', \delta_2', \delta_3' \ldots$ etc be the displacement corresponding to second system of forces.

Maxwell Reciprocal theoram states that first system acting through the corresponding displacement produced by the second system of forces do the same work as that of the 2nd system of forces acting through the displacement produced by the first system of forces.

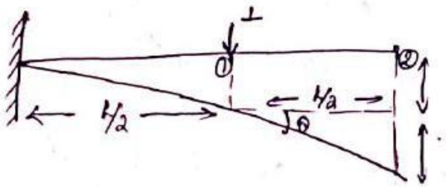
In otherwords, the workdone by the first system of forces on the second set of displacement must be equal to the workdone by the second system of forces on the third system of forces.

$$\left.\begin{array}{l} 1^{st}\text{ set} \\ 2^{nd}\text{ set} \end{array}\right\} \begin{array}{l} P_1, P_2, P_3 \\ P_1', P_2', P_3' \end{array} \qquad \begin{array}{l} \delta_1, \delta_2, \delta_3 \\ \delta_1', \delta_2', \delta_3' \end{array}$$

* find the transverse deflection at point 1 due to a load P at point ② of the cantleaver shown in the figure.



Inorder to solve this problem. Let us assume that there is a unit load acting at point ①



We know for a cantleaver beam carrying a load at free end. has a deflection and a slope of y & θ.

$$y = \frac{PL^3}{3EI} \qquad \theta = \frac{PL^2}{2EI}$$

$P = 1$ & $L = L/2$ at a point 1.

$$y = \frac{1 \times L^3}{24EI} = \frac{L^3}{24EI}$$

$$\theta = \frac{1 \times L^2}{8EI} = \frac{L^2}{8EI}$$

Now the deflection at point 2. due to unit load at 1. is given by

$$\delta_2 = P_1 a_{21} = a_{21}$$

a_{21} = deflection at point ① + deflection due to θ.

$$\delta_2 = a_{21} = \frac{L^3}{24EI} + \theta \times L/2$$

$$= \frac{L^3}{24EI} + \frac{\theta L}{2}$$

$$= \frac{L^3}{24EI} + \frac{L^2}{8EI} \times L/2$$

$$= \frac{L^3}{24EI} + \frac{L^3}{16EI}$$

$$= \frac{5L^3}{}$$

ie $a_{21} = \frac{5L^3}{48EI}$

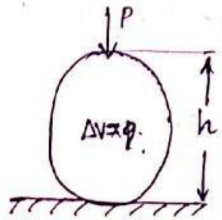
By Reciprocal theoram $a_{21} = a_{12}$

Hence $a_{12} = \frac{5L^3}{48EI}$.

we replace this unit load with a point load P at 2. Then deflection at ① due to this p.l is given by $\frac{5L^3}{48EI} \times P$.

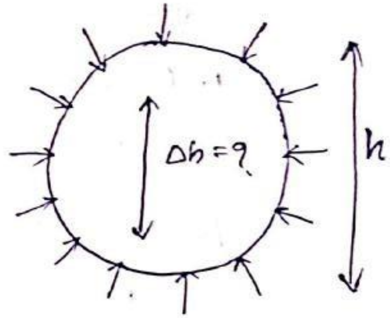
$$a_{12} = \frac{5PL^3}{48EI}$$

★ Determine the change in volume of an elastic body subjected to a compressive force as shown in fig. Take E & ν as elastic constants



The hydrostatic path of the stress tensor corresponds to the volume change of the body.

Let 'σ' be the magnitude of hydrostatic stress as shown in figure.



The strain is given by

$$\frac{\Delta h}{h} = \frac{1}{E}\left[\sigma - \nu(\sigma+\sigma)\right]$$

$$\Delta h = \frac{\sigma}{E}(1-2\nu) \times h$$

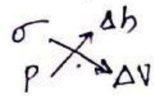
By reciprocating theoram,

$$P\Delta h = \sigma \times \Delta V$$

$$\Delta V = \frac{P \Delta h}{\sigma}$$

$$= \frac{P}{\sigma} \times \frac{\sigma}{E}(1-2\nu) \times h$$

$$= \frac{P}{E}(1-2\nu) h$$



★ Method of Virtual work of Minimum potential energy theoram

The method of virtual work or principal of virtual work is the most versatile method available for computing in the elastic unknown deflection of structure or unknown forces on the structures. This principal states that for a system, rigid or elastic in equilibrium, the sum of workdone by the external and internal (reactive) forces on a set of virtual displacements is zero.

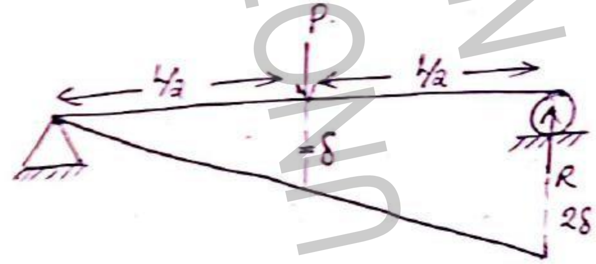
There are two principals of virtual work

1. Method of virtual displacement
2. " " " forces.

1. Method of virtual displacement

It is used to calculate the unknown forces.

Consider a rigid beam loaded at its centre by a concentrated load 'P' as shown in fig.

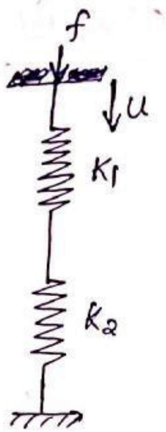


Since the beam is rigid and in order to satisfy the geometry, if 'δ' is the imaginary displacement at the centre, then the displacement at the end will be '2δ'.

$$\frac{l/2}{\delta} = \frac{l}{x}$$

$$x = 2\delta$$

According to method of virtual displacement,



The external workdone, $= f \times u$

Internal workdone $= fx_1 + fx_2$

$$= f\left(\frac{P}{K_1} + \frac{P}{K_2}\right).$$

Now the internal workdone will be equal to internal workdone

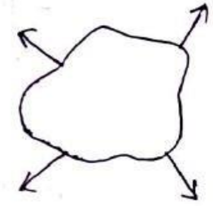
$$fu = f\left(\frac{P}{K_1} + \frac{P}{K_2}\right)$$

$$u = P\left(\frac{1}{K_1} + \frac{1}{K_2}\right)$$

* Minimum potential energy principal.

Consider an elastic-body subjected to external forces

X, Y, Z, S_x, S_y, S_z and $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz}$ are the corresponding reactive stresses.



Let $\delta u, \delta v$ & δw be the virtual displacement applied over the whole body. The virtual displacement produce virtual strain,

$$\delta \varepsilon_{xx} = \frac{\partial(\delta u)}{\partial x} \qquad \delta \gamma_{xy} = \frac{\partial(\delta v)}{\partial x} + \frac{\partial(\delta u)}{\partial y}$$

$$\delta \varepsilon_{yy} = \frac{\partial(\delta v)}{\partial y} \qquad \delta \gamma_{xz} = \frac{\partial(\delta w)}{\partial x} + \frac{\partial(\delta u)}{\partial z}$$

$$\delta \varepsilon_{zz} = \frac{\partial(\delta w)}{\partial z} \qquad \delta \gamma_{yz} = \frac{\partial(\delta w)}{\partial y} + \frac{\partial(\delta v)}{\partial z}$$

Now the total external work-done by the system will be

$$\int_V [(X \times \delta u) + (Y \times \delta v) + (Z \times \delta w)]\, dv + \int_S (S_x \times \delta u + S_y \times \delta v + S_z \times \delta w) \times ds.$$

Now the internal workdone is given by

$$= -\int_V (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz})\, dv.$$

By method of virtual work

external work + internal work $= 0$

$$\int_V [(X \times \delta u) + (Y \times \delta v) + (Z \times \delta w)]\, dv + \int_S (S_x \times \delta u + S_y \times \delta v + S_z \times \delta w)\, ds$$
$$+ \int -((\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz})\, dv = 0$$

$$\int_V (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz})\,dv$$

$$- \int_V (x\delta u + y\delta v + z\delta w)\,dv - \int_S (S_x\delta u + S_y\delta v + S_z\delta w)\,ds = 0 \quad \text{— (a)}$$

Since 'δ' is a variational symbol,

$\delta(\sigma_{xx}\varepsilon_{xx}) = \delta\sigma_{xx}\,\varepsilon_{xx} + \sigma_{xx}\,\delta\varepsilon_{xx}$

here σ_{xx} will not varying until the external loads are not varying.

$\therefore$ We can write $\delta\sigma_{xx} = 0$ which implies

$\delta(\sigma_{xx}\varepsilon_{xx}) = \sigma_{xx}\,\delta\varepsilon_{xx}$.

lly,

$\delta(xu) = x\delta u \quad (\delta x = 0)$

$\delta(S_x u) = S_x\delta u \quad (\delta S_x = 0)$

equ-(a) $\Rightarrow$

$$\delta\Big[\underbrace{\int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\,dv}_{\text{strain energy (U)}}$$

$$- \underbrace{\Big(\int_V (xu + yv + zw)\,dv + \int_S (S_x u + S_y v + S_z w)\,ds}_{\text{workdone (W)}} = 0$$

$\delta[U - W] = 0$ —— (1)

Difference b/w total strain energy of the system and the workdone by external forces is known as potential energy (π)

equ-(1) can be rewritten as

$$\boxed{\delta\pi = 0}$$

For stable equilibrium $\delta\pi > 0$.

This implies that potential energy function is maximum or minimum.

For stable equilibrium potential energy must be minimum.

This principal is known as minimum potential energy principal.

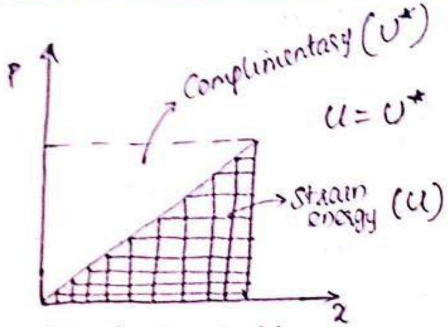
The equilibrium displacement of an elastic solid under the action of loads or a system of load ~~energy~~ is the one having the minimum potential energy.

★ ~~Stiffness Coefficent~~ (K_{ij}) Stiffness Coefficient (K_{ij})

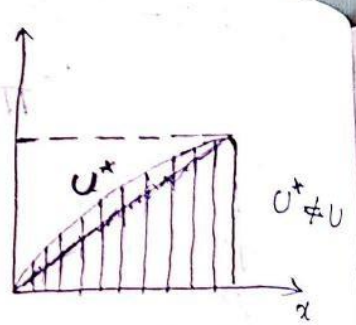
It is the net force acting at the i^{th} position or point due to a unit deflection or displacement at 'j'th position or point.

★ Complimentary energy Complementary

If a body is strained from an initial unstrained condition and final strained condition. The amount of workdone on it, is the total energy absorb or total strain energy.



(For linear elastic - Solids).



(For non linear elastic Solids)

The area enclosed by the load displacement diagram with respect to the deformation line is known as the strain energy (U). The area enclosed by the load displacement diagram with respect to the load line is termed as Complimentary energy (U^*).

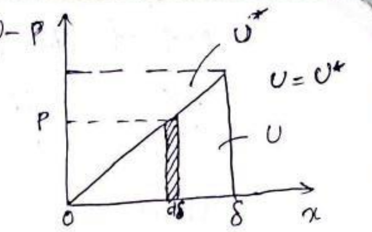
For linear elastic solid - variation are linearly and hence both areas are same.

Strain energy, U = Complimentary energy U^*

For a non linear elastic solids the variation is non linear in nature. Both the areas enclosed are different.

$U^* \neq U$.

Consider an elemental strip as shown in fig.



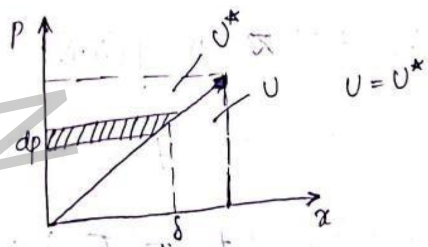
From the figure,

Area of the elemental strip,

$du = P d\delta$.

Total area, $U = \int du$

$$\boxed{U = \int_0^{\delta} P d\delta}$$



$U = U^*$

~~Corres~~

area, $du^* = dp.\delta$

$U^* = \int du^*$

$$\boxed{U^* = \int_0^{P} \delta . dp.}$$

Complimentary energy Principal

Consider an elastic solid subjected to body forces X, Y, Z. and surface traction S_x, S_y, S_z. Let $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz}$ be the stress developed

Method of Virtual forces used to Complimentary energy principal: Let $\delta x, \delta y, \delta z$ are the Virtual body forces $\delta S_x, \delta S_y, \delta S_z$ are the Virtual surface tractions. These Virtual forces will develope the virtual stresses is given by $\delta\sigma_{xx}, \delta\sigma_{yy}, \delta\sigma_{zz}, \delta\tau_{xy}, \delta\tau_{xz}, \delta\tau_{yz}$. Let u, v, w be the displacement produced in the system. The external work produced in the System.

$$\int_V (u\delta x + v\delta y + w\delta z)\,dv + \int_S (u\cdot\delta S_x + v\delta S_y + w\delta S_z)\,ds$$

The internal workdone produced in the system

$$-\int_V (\varepsilon_{xx}\delta\sigma_{xx} + \varepsilon_{yy}\delta\sigma_{yy} + \gamma_{yz}\delta\tau_{yz})\,dv = 0$$

Now by method of Virtual work.

external work + internal work = 0

$$\int_V (u\delta x + v\delta y + w\delta z)\,dv + \int_S (u\delta S_x + v\delta S_y + w\delta S_z)\,ds.$$
$$-\int_V (\varepsilon_{xx}\delta\sigma_{xx} + \varepsilon_{yy}\delta\sigma_{yy} + \gamma_{yz}\delta\tau_{yz}) = 0$$

$$\delta\Big[\int_V (\sigma_{xx}\varepsilon_{xx} + \varepsilon_{yy}\sigma_{yy} + \gamma_{yz}\tau_{yz})\,dv - \Big(\int_V (ux + vy + zw)\,dv + \int_S (S_x u + S_y v + S_z w)\,ds\Big]= 0$$

$$\delta[U^* - W] = 0.$$

$$\boxed{\delta\pi^* = 0}$$

π^* is known as Complimentary energy

Here also for a stable equilibrium the Complimentary PE must be minimum and this principal is known as Complimentary energy principal

Castiglrano's 2nd theorem

It states that the

Any particular force is the displacement of the point of application of that corresponding force in the direction of its line of action.

ie, $\delta_i = \frac{\partial U}{\partial P_i}$

Proof

Consider an elastic body subjected to concentrated loads $P_1, P_2, P_3, \ldots\ldots P_n$ and the corresponding displacements are $\delta_1, \delta_2, \delta_3 \ldots$ The total strain energy of the system is given by

$$U = \frac{1}{2}\left[P_1\delta_1 + P_2\delta_2 + \ldots\ldots P_n\delta_n\right] \quad \text{——(1)}$$

Now, we know.

$$\delta_1 = a_{11}P_1 + a_{12}P_2 + \ldots\ldots\ldots a_{1n}P_n$$

$$\delta_2 = a_{21}P_1 + a_{22}P_2 + \ldots\ldots\ldots a_{2n}P_n$$

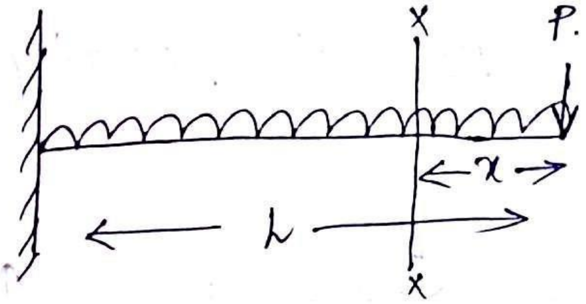
$$\vdots$$

$$\delta_n = a_{n1}P_1 + a_{n2}P_2 + \ldots\ldots\ldots a_{nn}P_n$$

Substitute these values of 'δ' in equ-(1)

$$U = \frac{1}{2}\Big[P_1(a_{11}P_1 + \ldots\ldots a_{1n}P_n) + P_2(a_{21}P_1 + \ldots\ldots a_{2n}P_n)$$
$$+ \ldots\ldots P_n(a_{n1}P_1 + a_{n2}P_2 + \ldots\ldots a_{nn}P_n)\Big]$$

Ans



$$Mx = Px + Wx \times \frac{x}{2}$$
$$= Px + \frac{Wx^2}{2}$$

$$U = \int \frac{M^2}{2EI} dx$$

$$\delta = \frac{\partial u}{\partial p} = \frac{\partial}{\partial p}\left[\int \frac{M^2}{2EI} dx\right]$$

$$= \int \frac{1}{\cancel{2}EI} \cdot \cancel{2}M \cdot \frac{\partial M}{\partial p} dx.$$

$$= \frac{1}{EI}\int M \cdot \frac{\partial M}{\partial p} dx.$$

$$= \frac{1}{EI}\int_0^l \left(Px + \frac{Wx^2}{2}\right) x \cdot dx$$

$$= \frac{1}{EI}\left[\frac{Px^3}{3} + \frac{Wx^4}{8}\right]_0^l$$

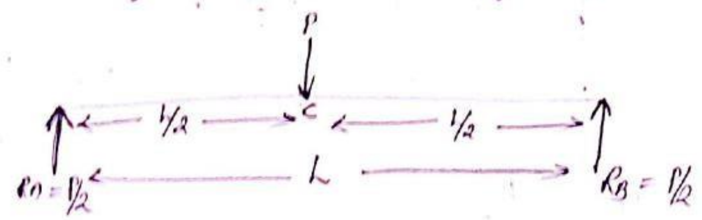
$$= \frac{1}{EI}\left[\frac{PL^3}{3} + \frac{WL^4}{8}\right]$$

For finding the deflection due to UDL alone

$P = 0.$

$$\delta = \frac{WL^4}{8EI}$$

* find the central deflection of a simply supported beam carrying a point load at its centre. Assume uniform flexural rigidity.



$R_A + R_B = P$

$\Sigma M_A = 0$

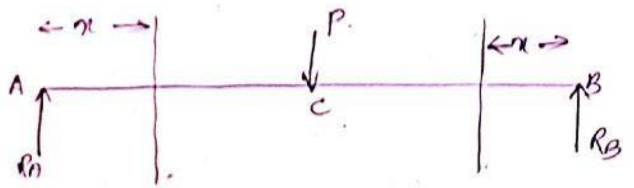
$P \times L/2 - R_B \times L = 0.$

$\frac{PL}{2} - R_B L = 0$

$R_B L = \frac{PL}{2}$

$R_B = P/2$

$\therefore R_A = P/2$



$U = \int \frac{M^2}{2EI}$

$M_A = 0.$

$R_A = P/2 \qquad R_B = P/2$

AC.

$M_x = R_A \times x.$

$= P/2\, x.$

$U = \int \frac{M^2}{2EI} dx$

$= 2 \int_0^{L/2} \frac{M^2}{2EI} dx.$

$= \frac{1}{EI} \int_0^{L/2} \left(\frac{Px}{2}\right)^2 dx$

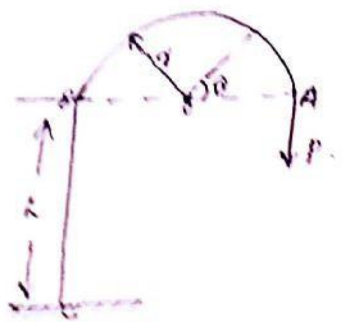
$= \frac{1}{EI} \int_0^{L/2} \frac{P^2 x^2}{4} dx$

$= \frac{1}{EI} \left[\frac{P^2 x^3}{12}\right]_0^{L/2}$

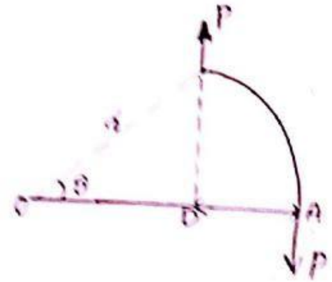
$= \frac{1}{EI} \left[\frac{P^2}{12} \frac{L^3}{8}\right]$

$= \frac{P^2 L^3}{96 EI}$

* For the structure shown in fig. calculate the vertical deflection at a point A by considering bending energy alone.



Ans:



The given structure can be divided into two: one vertical member CB and other curved member BA. Consider the vertical member CB,

Here the load acting at a distance 2r.

$M_x = 2Pr$

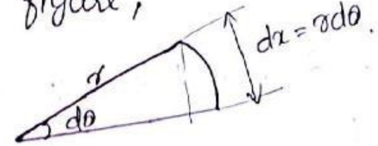
Now considering the curved member BA. Consider a small elemental portion. Here the distance b/w applied load 'P' and section is DA and is equated as

$DA = OA - OD$

$= r - r\cos\theta$

$= r(1-\cos\theta)$

Now from the figure,



Bending moment $M_{AB}/r = Pr(1-\cos\theta)$ [Curved surface]

Total energy of the section, $U = U_{CB} + U_{AB}$

$$U = \int_0^L \frac{M^2}{2EI}\,dy + \int_0^\pi \frac{M^2}{2EI}\, r\,d\theta$$

$$= \int_0^L \frac{4P^2r^2}{2EI}\,dy + \int_0^\pi \frac{P^2r^2(1-\cos\theta)^2}{2EI}\, r\,d\theta$$

$$= \frac{4P^2r^2}{2EI}\Big[y\Big]_0^L + \int_0^\pi \frac{P^2r^2}{2EI}\left(1^2 - 2\cos\theta + \cos^2\theta\right) r\,d\theta$$

$$= \frac{4P^2r^2}{2EI}\times L + \frac{P^2r^3}{2EI}\int_0^\pi \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \frac{4P^2r^2}{2EI}\times L + \frac{P^2r^3}{2EI}\left[\theta - 2\sin\theta\right]_0^\pi + \int_0^\pi \frac{1+\cos2\theta}{2}\,d\theta$$

$$= \frac{4P^2r^2}{2EI}\times L + \frac{P^2r^3}{2EI}\left[\theta - 2\sin\theta\right]_0^\pi + \left[\frac{\theta}{2} + \frac{\sin2\theta}{4}\right]_0^\pi$$

$$= \frac{4P^2 r^2}{2EI} \times L + \frac{P^2 r^3}{2EI}\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= \frac{4P^2 r^2}{2EI} \times L + \frac{P^2 r^3}{2EI}\left[\left(\pi - 2\times 0 + \frac{\pi}{2} + \frac{-0}{4}\right) - (0+0-0)\right]$$

$$= \frac{4P^2 r^2}{2EI} \times L + \frac{P^2 r^3}{2EI}\left[\pi + \frac{\pi}{2}\right]$$

$$= \frac{4P^2 r^2}{2EI} \times L + \frac{P^2 r^3}{2EI} \times \frac{3\pi}{2}$$

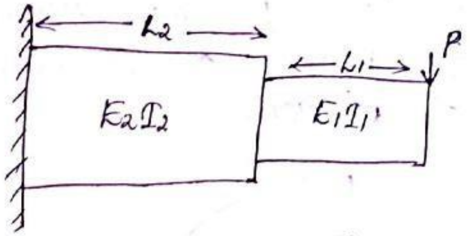
$$= \frac{4P^2 r^2 \times L}{2EI} + \frac{3P^2 r^3 \pi}{4EI}$$

$$U = \frac{3\pi P^2 r^3 + 8P^2 r^2 L}{4EI}$$

$$\delta_A = \frac{\partial U}{\partial P} = \frac{1}{4EI}\left[6\pi P r^3 + 16 P r^2 L\right]$$

$$= \frac{1}{2EI}\left[3\pi P r^3 + 8 P r^2 L\right]$$

★ For the cantileaver shown in fig. find the transverse deflection at the free end by neglecting the shear deformation.



$$M = Px.$$

Total energy, $U = U_1 + U_2$

$$= \int_0^{L_1} \frac{M^2}{2E_1I_1} dx + \int_{L_1}^{L_1+L_2} \frac{M^2}{2E_2I_2}$$

$$= \int_0^{L_1} \frac{(Px)^2}{2E_1I_1} dx + \int_{L_1}^{L_1+L_2} \frac{(Px)^2}{2E_2I_2}$$

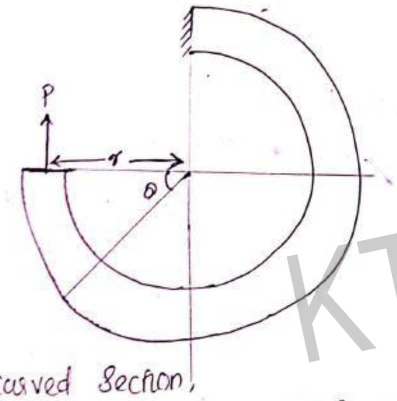
$$= \frac{P^2}{2E_1I_1}\left[\frac{x^3}{3}\right]_0^{L_1} + \frac{P^2}{2E_2I_2}\left[\frac{x^3}{3}\right]_{L_1}^{L_1+L_2}$$

$$= \frac{P^2}{2E_1I_1}\left[\frac{L_1^3}{3}\right] + \frac{P^2}{2E_2I_2}\left[\frac{(L_1+L_2)^3}{3} - \frac{(L_1)^3}{3}\right]$$

$$= \frac{P^2}{2E_1I_1}\left[\frac{L_1^3}{3}\right] + \frac{P^2}{2E_2I_2}\left[\frac{(L_1+L_2)^3 - (L_1)^3}{3}\right]$$

$$= \frac{P^2}{6}\left[\frac{L_1^3}{E_1I_1} + \frac{(L_1+L_2)^3 - (L_1)^3}{E_2I_2}\right]$$

* The curved beam shown in fig has 80 mm square section and the radious of curvature $r = 65$ mm. The beam is made of steel having young's modulus 200 GPa, poisons ratio .29. If the applied load $P = 6$ kN. Determine the deflection at the free end of the curved beam in the vertical directions.



for curved section,

Bending moment, $M = Pr(1-\cos\theta)$

$$\delta = \frac{\partial U}{\partial P}$$

$$U = \int \frac{M^2}{2EI}\, dx \qquad dx = r\,d\theta$$

$$= \int \frac{M^2}{2EI}\, r\,d\theta.$$

$$\delta = \frac{\partial}{\partial P}\left[\int_0^{3\pi/2} \frac{M^2}{2EI}\, r\,d\theta\right]$$

$$= \int_0^{3\pi/2} \frac{M}{EI}\frac{\partial M}{\partial P}\, r\,d\theta$$

$$= \int_0^{3\pi/2} \frac{Pr(1-\cos\theta)}{EI} \times r(1-\cos\theta) \times r\,d\theta.$$

$$= \frac{Pr^3}{EI}\int_0^{3\pi/2} (1-\cos\theta)^2\, d\theta$$

$$= \frac{Pr^3}{EI}\int_0^{3\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta.$$

$$= \frac{Pr^3}{EI}\left[\theta - 2\sin\theta\right]_0^{3\pi/2} + \int_0^{3\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta.$$

$$= \frac{Pr^3}{EI}\left[\theta - 2\sin\theta + \theta/2 + \frac{\sin 2\theta}{4}\right]_0^{3\pi/2}$$

$$= \frac{Pr^3}{EI}\left[\left(\frac{3\pi}{2} - 2\sin\frac{3\pi}{2} + \frac{3\pi}{4} + \frac{\sin 2\times\frac{3\pi}{2}}{4}\right) - (0 - 0 + 0 - 0)\right]$$

$$= \frac{Pr^3}{EI}\left[\frac{3\pi}{2} + \frac{3\pi}{4} + 2\right]$$

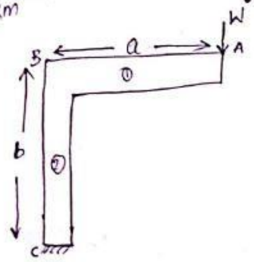
$$\delta = \frac{Pr^3}{EI}\left[\frac{9\pi}{4} + 2\right]$$

$$\delta = \frac{6\times10^3 \times 65^3}{200\times10^3 \times \frac{30^4}{12}}\left[\frac{9\pi}{4}+2\right]$$

$$= 1.1062 \text{ mm}$$

$$I = \frac{bd^3}{12} = \frac{30\times30^3}{12} = \frac{30^4}{12}$$

* Find the Vertical deflection of the L-section as shown figure. Assume L section is uniform



$U = U_{BC} + U_{AB}$

$$U_{AB} = \int_0^a \frac{(Wx)^2}{2AE}dx$$

$M_{AB} = Wx$

$M_{CB} = Wa$

$$= \frac{W^2}{2AE}\left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{W^2}{2AE}\left(\frac{a^3}{3}-0\right)$$

$$= \frac{W^2a^3}{6AE}$$

$$U_{BC} = \int_0^b \frac{M^2}{2AE}dx$$

$$= \int_0^b \frac{(Wa)^2}{2AE}dx$$

$$= \frac{W^2a^2}{2AE}(x)_0^b$$

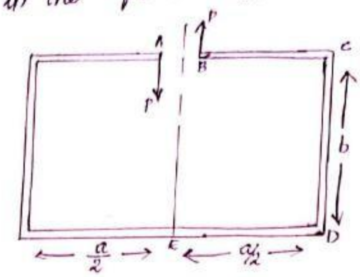
$$= \frac{W^2a^2}{2AE}(b)$$

$$U = U_{AB} + U_{BC}$$

$$= \frac{W^2x^2}{2AE}\times a + \frac{W^2a^2}{2AE}\times b$$

$$= \frac{W^2}{2AE}(ax^2 + ba^2)$$

* find out the relative displacements of points A & B in the frame shown in figure.



Consider the member BC & ED.

Bending moment is varying linearly with respect to the distance. ie

$M = Px$.

Consider the member CD.

Here the Bending moment is constant because each point in the member CD is at a distance of $a/2$ from the load p

$\therefore M_{CD} = P \times \frac{a}{2}$

Now the ~~[illegible]~~ total strain energy on the system is given by

$U = U_{BC} + U_{ED} + U_{CD}$

$= 2U_{BC} + U_{CD}$

$= 2\int_0 \frac{p^2x^2}{2EI}dx + \int_0 \frac{\frac{p^2a^2}{4}}{2EI}dy$

$= 2\frac{p^2}{2EI}\left[\frac{x^3}{3}\right]_0^{a/2} + \frac{p^2a^2}{8EI}\left[y\right]_0^b$

$= 2\cdot\frac{p^2}{2EI}\left[\frac{(\frac{a}{2})^3}{3} - 0\right] + \frac{p^2a^2}{8EI}\left[b-0\right]$

$= \cancel{2}\cdot\frac{p^2}{\cancel{2}EI} \times \frac{a^3}{24} + \frac{p^2a^2b}{8EI}$

$= \frac{p^2a^3}{24EI} + \frac{p^2a^2b}{8EI}$

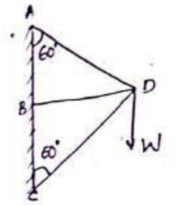
Deflection at the point $B = \frac{\partial U}{\partial p}$

$= \frac{\cancel{2}pa^3}{\cancel{24}EI} + \frac{\cancel{2}pa^2b}{\cancel{8}EI}$

$= \frac{pa^3}{12EI} + \frac{pa^2b}{4EI}$

Since two parts are identical, the magnitude of deflection of A is equal to the magnitude of deflection at B. But the direction is downward.

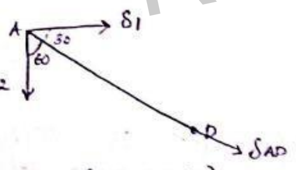
★ AD, BD & CD are three elastic members connected by smooth pins. All the members have same area and AD = CD = 1 m. find out the deflection under the load W.



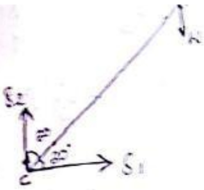
AD = CD = 1 m

we are given with the horizontal deflection δ_1 & vertical deflection δ_2. we need to find the axial deflection of each member.

Inorder to find the axial deflection draw the free body diagrams of each members



$AD = \delta_1 \cos 30 + (-\delta_2 \cos 60)$



$CD = \delta_1 \cos 30 + \delta_2 \cos 60$

The total strain energy is given by

$U = \frac{1}{2} \times$ stress $\times$ strain $\times$ volume

$= \frac{1}{2} \times E \times \frac{\delta}{L} \times \frac{\delta}{L} \times AL$

$$U = \frac{E\delta^2 A}{2L}$$

$U = U_{CD} + U_{AD} + U_{BD}$

$$= \frac{EA}{2}\left[\frac{(\delta_2 \cos 60 + \delta_1 \cos 30)^2}{1} + \frac{\delta_1^2}{\sin 60} + \frac{(-\delta_2 \cos 60 + \delta_1 \cos 30)^2}{1}\right]$$

$$= \frac{EA}{2}\left[\frac{(.5\delta_2 + .866\delta_1)^2}{1} + \frac{\delta_1^2}{\sqrt{3}/2} + \frac{(-\frac{1}{2}\delta_2 + \frac{1}{2}\delta_1)^2}{1}\right]$$

$$= \frac{EA}{2}\left[\frac{(\frac{1}{2}\delta_2)^2 + 2 \times \frac{1}{2}\delta_2 \times \frac{\sqrt{3}}{2}\delta_1 + (\frac{\sqrt{3}}{2}\delta_1)^2}{1} + \frac{\delta_1^2}{\sqrt{3}/2} + (-\frac{1}{2}\delta_2)^2 + 2 \times -\frac{1}{2}\delta_2 \times \frac{1}{2}\delta_1 + (\frac{1}{2}\delta_1)^2\right]$$

$$= \frac{EA}{2}\left[\frac{1}{4}\delta_2^2 + \frac{\sqrt{3}}{2}\delta_1\delta_2 + \frac{3}{4}\delta_1^2 + \frac{\delta_1^2}{\sqrt{3}/2} + \frac{1}{4}\delta_2^2 - \frac{1}{2}\delta_1\delta_2 + \frac{1}{4}\delta_1^2\right]$$

$$= \frac{EA}{2}\left[\frac{2\delta_2^2 + 6\delta_1^2}{4} + \frac{2\delta_1^2}{\sqrt{3}}\right]$$

$$= AE\left[\frac{6\delta_1^2 + 2\delta_2^2}{8} + \frac{\delta_1^2}{\sqrt{3}}\right]$$

By Castigliano's first theoram

$\frac{\partial U}{\partial \delta_1} = 0$ (Since there is no load along the direction of δ_1)

$\left(\frac{\partial U}{\partial \delta_2}\right) = -W$ (Since W is in the opposite direction of δ_2)

$$\frac{\partial U}{\partial \delta_1} = AE\left[\frac{12\delta_1}{8} + \frac{2\delta_1}{\sqrt{3}}\right] = 0$$

$$EA\,\delta_1\left[\frac{12}{8} + \frac{2}{\sqrt{3}}\right] = 0$$

$$\delta_1 = 0$$

Now strain energy,

$$U = AE\left[\frac{\delta_2^2}{4}\right]$$

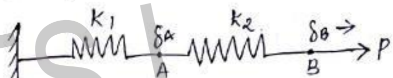
$$\frac{\partial U}{\partial \delta_2} = AE\left[\frac{4\delta_2}{8}\right] = -W$$

$$\delta_2 = \frac{-2W}{AE}$$

$$U = U_{CD} + U_{AD} + U_{BD}$$

$=$

* Apply the principal of minimum potential energy desolue deflection at 'B' for the spring system shown in figure



~~external workdone~~

$$U = U_1 + U_2$$

$$= \frac{1}{2}K_1\delta_A^2 + \frac{1}{2}K_2(\delta_B - \delta_A)^2$$

Workdone, $W = P\delta_B$

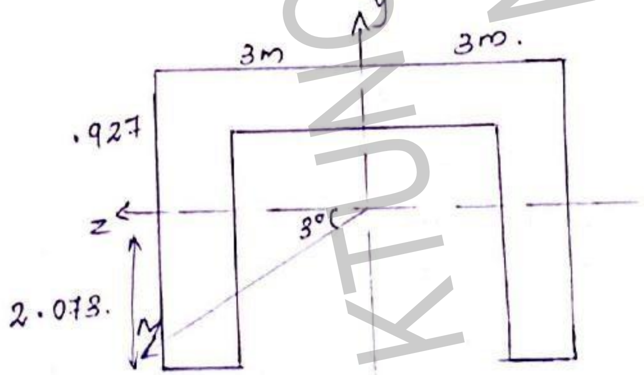
Minimum potential energy,

$$\Pi = U - W = \frac{1}{2}K_1\delta_A^2 + \frac{1}{2}K_2(\delta_B - \delta_A)^2 - P\delta_B$$

Applying Minimum potential energy theoram

$$\delta_B = \frac{P(K_1+K_2)}{K_1K_2}$$

* Due to the load misallingment, the Bending-moment acting on the channel section is inclined at an angle 3° with respect to the z-axis as shown in fig. If the allowable flexural stress for this Beam is 16 N/m², what is the maximum moment that can be applied. For the chanel section this assume that the product moment of inertia is zero $I_{zz} = 8.22\ m^4$, $I_{yy} = 26\ m^4$



$$M_z = M\cos\alpha = M\cos 3$$

$$= .998M\ Nm$$

$$M_y = M\sin\alpha = M\sin 3$$

$$= .052M\ Nm.$$

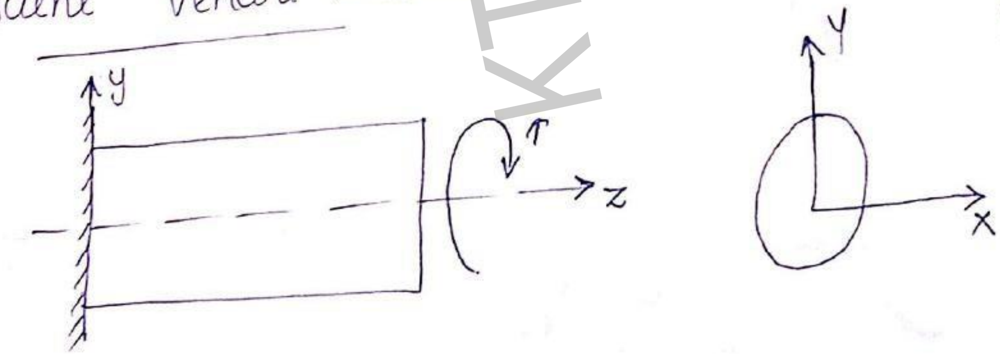
Non- Circular / Prismatic bar

Generally the Circular Cross-Section has smooth boundaries. Where as for a non circular sections like rectangle or triangle the boundaries have sharp edges. Therefore when we apply a twisting moment to a non circular shaft. The cross section may not remain as such after twisting, there is a deformation with in the plane and is termed as plane deformation. This out of plane deformation is represented by a function called torsion function or warping function.

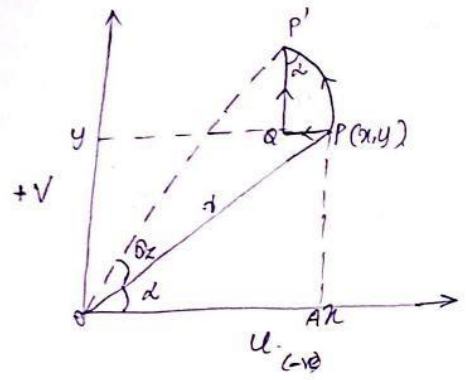
There are two methods for analysis torsion of non circular shaft, they are

1. Saint Venant Method
2. Prandl's method

① Saint Venant Method



Consider a shaft of uniform cross section throughout the length as shown in fig. under the action of torque, deformation of shaft will takes place and these deformation give rise to stresses at each and every point on cross sections. Consider a typical cross section in which let a point 'p' moves to 'p'' under the action of 'T'.



This displacement is going to give rise some motion in x & y directions. Here the typical cross section will n't remain plane after applying a twisting moment or torque. Here the cross section is uniform throughout the length of the shaft [Hence we can assume that the deformation will remain the same along the length of the shaft. Consider that the section deformation, typically known as warping, in the z-direction will be simply a fn of its position along. It does not depend upon where it is located along the length of the shaft (z-axis). ie warping is a fn of x & y and is independent of z.]

Let 'θ' = angle of twist per unit length.

Here the section is considered at a distance of z and hence total angle of twist = θz.

Here $OP = r$

$$PP' = \text{radious} \times \text{angle}$$

$$= r \times \theta z$$

$$= r\theta z$$

from ΔOPA.

$$\sin\alpha = y/r$$

$$\cos\alpha = \frac{x}{r}$$

Now consider the $\Delta PQP'$.

After twisting the point 'p' moves to 'p'' where the position xy is changed to x'y'. It indicates that the point 'p' has a displacement along x-direction represented by PQ and displacement along y-direction represented by QP'.

$$\therefore \quad PQ = u = -PP'\sin\alpha$$

$$= -r\theta z \times y/r$$

$$\boxed{u = -\theta z y}$$

Similarly $QP' = v = PP' \cos\alpha$

$$= r\theta z \times \frac{x}{r}$$

$$\boxed{v = \theta z x}$$

~~can be~~ ~~expre~~ Deformation along z-direction can be represented as,

$$w = \theta\, \psi(x,y)$$

where $\psi(x,y)$ is known as the warping function or torsion function, which is independent of z.

By strain displacement relation

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial(-\theta z y)}{\partial x}$$

$$= \underline{\underline{0}}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial(\theta z x)}{\partial y}$$

$$= \underline{\underline{0}}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = \frac{\partial(\theta \psi(x,y))}{\partial z}$$

$$= \underline{\underline{0}}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$= \frac{\partial(-\theta z y)}{\partial y} + \frac{\partial(\theta z x)}{\partial x}$$

$$= -\theta z + \theta z.$$

$$= \underline{\underline{0}}.$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}.$$

$$= \theta \times \frac{\partial \psi}{\partial x} + -\theta y$$

$$= \underline{\underline{\theta\left[\frac{\partial \psi}{\partial x} - y\right]}}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$= \theta x + \theta \times \frac{\partial \psi}{\partial y}$$

$$= \theta\left[x + \frac{\partial \psi}{\partial y}\right)$$

The above equation there is no normal strain value. ∴ There is no normal stresses. But there exist two shear strain components γ_{xz} & γ_{yz} and hence there will be shear stress components given by

$$\tau_{xz} = G\gamma_{xz}$$
$$= G\theta\left(\frac{\partial\psi}{\partial x} - y\right) \quad \text{--- (1)}$$

$$\tau_{yz} = G\gamma_{yz}$$
$$= G\theta\left(\frac{\partial\psi}{\partial y} + x\right) \quad \text{--- (2)}$$

Boundary conditions

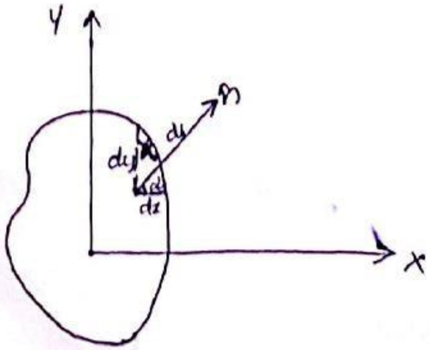
S_x, S_y, S_z are the surface traction component then by Cauchy's stress equation we have.

$$S_x = \sigma_{xx} n_x + \tau_{xy} n_y + \tau_{xz} n_z$$

$$S_y = \tau_{xy} n_x + \sigma_{yy} n_y + \tau_{yz} n_z.$$

$$S_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_{zz} n_z$$

Let 'n' be the unit normal of surface show in fig.

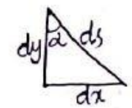


We know $n = n_x\hat{i} + n_y\hat{j} + n_z k$

where n_x, n_y, n_z are direction cosines along x, y & z directions.

from fig.

$$n_x = \cos\alpha = \frac{dy}{ds}$$



$$n_y = \cos(90-\alpha) = \sin\alpha = -\frac{dx}{ds}.$$ (-ve sign indicates that displacement is along -ve x axis)

$$n_z = \cos 90 = 0$$

We have the Cauchy's equation

$$T_{nx} = \sigma_{xx} n_x + \tau_{xy} n_y + \tau_{xz} n_z.$$

$$T_{ny} = \tau_{xy} n_x + \sigma_{yy} n_y + \tau_{yz} n_z$$

$$T_{nz} = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_{zz} n_z.$$

Substituting the values of stresses.

$$T_{nx} = \tau_{xz} n_z$$ (σ_{xx} & τ_{xy} zero ?)

$$= \tau_{xz} \times 0 = \underline{\underline{0}}$$

$$T_{ny} = 0 + 0 + \tau_{yz} n_z.$$

$$= \tau_{yz} \times 0 = \underline{\underline{0}}$$

$$T_{nz} = \underline{\underline{\tau_{xz} n_x + \tau_{yz} n_y}}$$

Here we are applying torque alone and hence there is move others forces on the boundary. ie their is no traction

Hence $T_{nz} = 0 \Rightarrow \tau_{xz} n_x + \tau_{yz} n_y = 0$

$$\left(G\theta\left[\frac{\partial \psi}{\partial x}-y\right]\frac{dy}{ds}\right)+\left(G\theta\left[\frac{\partial \psi}{\partial y}+x\right]\cdot\frac{-dx}{ds}\right)=0$$

$$\boxed{\left(\frac{\partial \psi}{\partial x}-y\right)\frac{dy}{ds}-\left(\frac{\partial \psi}{\partial y}+x\right)\frac{dx}{ds}=0}$$

* Stress tensor.

$$\sigma_{xx}=\sigma_{yy}=\sigma_{zz}=\tau_{xy}=0$$

$$\tau_{xz}=G\theta\left(\frac{\partial \psi}{\partial x}-y\right)$$

$$\tau_{yz}=G\theta\left[\frac{\partial \psi}{\partial y}+x\right]$$

$$\sigma_{ij}=\begin{bmatrix}0 & 0 & G\theta\left(\frac{\partial \psi}{\partial x}-y\right)\\ 0 & 0 & G\theta\left(\frac{\partial \psi}{\partial y}+x\right)\\ G\theta\left(\frac{\partial \psi}{\partial x}-y\right) & G\theta\left(\frac{\partial \psi}{\partial y}+x\right) & 0\end{bmatrix}$$

* equilibrium equation

We have.

$$\left.\begin{aligned}\frac{\partial(\sigma_{xx})}{\partial x}+\frac{\partial(\tau_{xy})}{\partial y}+\frac{\partial(\tau_{xz})}{\partial z}&=0\\ \frac{\partial(\tau_{xy})}{\partial x}+\frac{\partial(\sigma_{yy})}{\partial y}+\frac{\partial(\tau_{yz})}{\partial z}&=0\\ \frac{\partial(\tau_{xz})}{\partial x}+\frac{\partial(\tau_{yz})}{\partial y}+\frac{\partial(\sigma_{zz})}{\partial z}&=0\end{aligned}\right\}$$

Assuming that there is no body force components.

Considering [illegible] equation

$\sigma_{xx}=0$, $\tau_{xy}=0$.

$$0+0+\frac{\partial}{\partial z}\left(G\theta\left(\frac{\partial \psi}{\partial x}-y\right)\right)=0$$

$$\underline{\underline{0=0}}$$

Satisfies equilibrium condition.

Considering second equation.

$\tau_{xy}=0$, $\sigma_{yy}=0$.

$$0+0+\frac{\partial}{\partial z}\left(G\theta\left(\frac{\partial \psi}{\partial y}+x\right)\right)=0$$

$$\underline{\underline{0=0}}$$

Satisfying equilibrium condition

Considering 3rd equation.

$$\frac{\partial\left[G\theta\left(\frac{\partial \psi}{\partial x}-y\right)\right]}{\partial x}+\frac{\partial\left[G\theta\left(\frac{\partial \psi}{\partial y}+x\right)\right]}{\partial y}+0=0$$

$$G\theta\left[\frac{\partial\left(\frac{\partial \psi}{\partial x}-y\right)}{\partial x}+\frac{\partial\left[\frac{\partial \psi}{\partial y}+x\right]}{\partial y}\right]=0$$

$$G\theta\left[\frac{\partial^2 \psi}{\partial x^2}+0+\frac{\partial^2 \psi}{\partial y^2}+0\right]=0$$

$$\left[\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}\right]=0$$

$$\underline{\underline{\nabla^2\psi=0}}$$

It is called Laplace equation. Barbic. fn in torsion satisfies the laplace equation. It is a boundary value problem and the Boundary Condition is

$$\left(\frac{\partial \phi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial \phi}{\partial x} + x\right)\frac{dx}{ds} = 0$$

* equation of torque in terms of Barbic - function.

Consider a small elemental strip of area 'dA' as shown in fig. The shear stress components are τ_{xz} & τ_{yz}. Corresponding to shear forces along x & y directions acting on small area are $\tau_{xz}\,dx\,dy$, $\tau_{yz}\,dx\,dy$. The moment of the shear forces are

$\tau_{xz} \cdot dx \cdot dy \times y$ (C.W)

$\tau_{yz}\, dx\, dy \times x$ (C.C.W)

Taking CCW moment are positive. The net moment will be $(\tau_{yz}\,dx\,dy \cdot x) - (\tau_{xz}\,dx\,dy \cdot y)$. The total resisting moment developing body will be

$$\iint (\tau_{yz}x - \tau_{xz}y)\,dx\,dy$$

At equilibrium this existing moment must be equal to the net torque applying.

ie

$$T = \iint (\tau_{yz}x - \tau_{xz}y)\,dx\,dy$$

$$T = \iint \left[xG\theta\left(\frac{\partial \phi}{\partial y} + x\right) - yG\theta\left(\frac{\partial \phi}{\partial x} - y\right)\right] dx\,dy$$

$$= G\theta \iint \left(x \cdot \frac{\partial \phi}{\partial y} + x^2 - y\frac{\partial \phi}{\partial x} + y^2\right) dx\,dy$$

$$= G\theta \iint \left[(x^2+y^2) + \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x}\right)\right] dx\,dy$$

$$T = G\theta J$$

where $\boxed{J = \iint (x^2+y^2) + \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x}\right) dx\,dy}$

Where the product 'GJ' is known as torsional rigidity

Approximate soln for circular bar using Saint Venant method

* Check wether the function $\psi = c$, is a possible warping function for the torsion of prismatic bar. Where c is a constant. If so, find out

1. Shape of the cross section
2. J - integral.
3. Angle of twist per unit length, for transmitting a torque 'T'.
4. Resultant stress variation.
5. Torsion equation

Ans.

$\psi = c$

$\frac{\partial \psi}{\partial x} = 0$ $\qquad$ $\frac{\partial \psi}{\partial y} = 0$

$\frac{\partial^2 \psi}{\partial x^2} = 0$ $\qquad$ $\frac{\partial^2 \psi}{\partial y^2} = 0$

$\nabla^2 \psi = 0$

$\therefore$ It is a possible warping fn. only if it satisfies $\nabla^2 \psi = 0$

(i) Shape : We have to defined boundary condition.

$$\left(\frac{\partial \psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial \psi}{\partial y} + x\right)\frac{dx}{ds} = 0$$

A circular shaft 10 cm dia & 1.1 m length is required to transmit power with a max angle of twist of 3°.

1. What is the max. possible load
2. What is the max. shear stress induced
3. Find the max. possible power transmitted for a speed of 1500 rpm.

Given that modulus of rigidity 8×10^4 MPa

Ans.

$d = 10\ cm = .1\ m$

$l = 1.1\ m.$

$\theta = 3^\circ = 3\times\frac{\pi}{180}$ rad

$G = 8\times10^4\ MPa = 80\times10^9\ N/m^2.$

$N = 1500$ rpm

From torsion equation.

$$\frac{T}{J} = \frac{G\theta}{L}.$$

$$T = \frac{G\theta}{L}\times J$$

$$J = \frac{\pi}{32}(.1)^4.$$

$$= 9.81\times10^{-6}\ N/m^2.$$

$$T = \frac{80\times10^9\times3\times\frac{\pi}{180}}{1.1}\times9.81\times10^{-6}$$

$= 3.738 \times 10^4 \text{ Nm}$

2. $\frac{\tau}{R} = \frac{G\theta}{L}$.

$\tau = \frac{G\theta}{L} \times R$.

At inner side, $r=0$, $\tau = 0$

outer side, $r = max = d/2$.

$$\tau_{max} = \frac{G\theta}{L} \times d/2.$$

$$= \frac{80 \times 10^9 \times 3 \times \frac{\pi}{180}}{1.1} \times 1/2$$

$$= 190.39 \times 10^6 \text{ N/m}^2$$

8. $P_{max} = T_{max} \times \omega$

$$= \frac{2\pi N T_{max}}{60}$$

$$= \frac{2\pi \times 1500 \times 3.738 \times 10^4}{60}$$

$$= 586.86 \times 10^4 \text{ W}$$

$$= 5868 \text{ KW}.$$

Approximate soln for elliptical bar using - Saint Venant's method

* Investigate the fn $\psi = Axy$, for the possibility as a warping fn, where A is a constant. If it is a possible soln, find out the shape of the cross section & also the J- integral.

1. $\psi = Axy$

$\frac{\partial \psi}{\partial x} = Ay$ $\qquad \frac{\partial \psi}{\partial y} = Ax$

$\frac{\partial^2 \psi}{\partial x^2} = 0.$ $\qquad \frac{\partial^2 \psi}{\partial x^2} = 0$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi$$
$$= 0$$

It is a warping function. It satisfies - laplace equation

2. $\left(\frac{\partial \psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial \psi}{\partial y} + x\right)\frac{dx}{ds} = 0$

$\left(\frac{\partial Axy}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial Axy}{\partial y} + x\right)\frac{dx}{ds} = 0$

$(Ay - y)\frac{dy}{ds} - (Ax + x)\frac{dx}{ds} = 0$

$y[1-A]\frac{dy}{ds} + x(1+A)\frac{dx}{ds} = 0$

MODULE-6

Prandtl stress fn method

The boundary condition obtained from Sant Venants method leads to complication then the order of the fn 'ψ' is higher. An alternative approach proposed by Prandtl - leads to a simpler boundary condition.

The assumptions in prandtl method are same as considered for Sanit Venants method.

These assumptions conclude that $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$ & the non zero stress components are - τ_{xz} and τ_{yz}.

Consider the equilibrium equations

$$\frac{\partial(\sigma_{xx})}{\partial x} + \frac{\partial(\tau_{xy})}{\partial y} + \frac{\partial(\tau_{xz})}{\partial z} = 0$$

$$\frac{\partial(\tau_{xy})}{\partial x} + \frac{\partial(\sigma_{yy})}{\partial y} + \frac{\partial(\tau_{yz})}{\partial z} = 0$$

$$\frac{\partial(\tau_{xz})}{\partial x} + \frac{\partial(\tau_{yz})}{\partial y} + \frac{\partial(\sigma_{zz})}{\partial z} = 0$$

Here the 1st & 2end equilibrium equations are satisfied exactly by the components of stress and to 3rd set of equilibrium equations satisfies to

$$\frac{\partial(\tau_{xz})}{\partial x} + \frac{\partial(\tau_{yz})}{\partial y} = 0$$

This equation can be satisfied exactly by selecting

$$\boxed{\tau_{xz} = \frac{\partial\phi}{\partial y} \quad \& \quad \tau_{yz} = -\frac{\partial\phi}{\partial x}}$$

where 'ϕ' is known as prandtl's stress - functions.

Strain Components using prandtl's stress - functions

we have the stress components

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0.$$

$$\tau_{xz} = \frac{\partial\phi}{\partial y}$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x}.$$

By hookes law

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} = 0.$$

$$\varepsilon_{yy} = \varepsilon_{zz} = 0.$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0$$

$$\sigma = E\varepsilon$$

$$\tau = G\gamma$$

$$\gamma_{xz} = \frac{1}{G}\tau_{xz} = \frac{1}{G}\frac{\partial\phi}{\partial y}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz} = \frac{-1}{G}\frac{\partial\phi}{\partial x}$$

Stress tensor is

$$\frac{2}{8C^3}$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \frac{\partial\phi}{\partial y} \\ 0 & 0 & -\frac{\partial\phi}{\partial x} \\ +\frac{\partial\phi}{\partial y} & -\frac{\partial\phi}{\partial x} & 0 \end{bmatrix}$$

Strain tensor is

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \frac{1}{G}\frac{\partial\phi}{\partial y} \\ 0 & 0 & -\frac{1}{G}\frac{\partial\phi}{\partial x} \\ \frac{1}{G}\frac{\partial\phi}{\partial y} & \frac{-1}{G}\frac{\partial\phi}{\partial x} & 0 \end{bmatrix}$$

* Equation of torque in terms of stress-functions

we have, $T = \iint (x \cdot \tau_{yz} - y \cdot \tau_{xz})\, dA$

$$= -\iint \left(x \cdot \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y}\right) dx\,dy$$

$$= -\left[\iint x \cdot \frac{\partial \phi}{\partial x}\, dx\,dy + \iint y \cdot \frac{\partial \phi}{\partial y}\, dx\,dy\right]$$

using integration by parts FIS − IDFIS

$$= -\left[\int (x\phi - \int 1 \cdot \phi \cdot dx)\, dy + \int \left[y\phi - \int 1 \cdot \phi\, dy\right] dx\right]$$

$$= -\left[\int x \cdot \phi\, dy - \int \phi\, dx\,dy + \int y\phi \cdot dx - \int 1 \cdot \phi\, dy\, dx\right]$$

$$= -\left[\int x\phi\, dy - \iint \phi\, dx\,dy + \int y\phi\, dx - \iint \phi\, dx\,dy\right]$$

$\because$ ϕ is constant around boundary, and the variation of 'ϕ' w.r.to x & y is zero and hence if we consider the terms $\int x\phi\, dy$ & $\int y\phi\, dx$ becomes zero

$$T = -\left[0 - \iint \phi\, dx\,dy + 0 - \iint \phi\, dx\,dy\right]$$

$$\boxed{T = 2\iint \phi\, dx\,dy}$$

* Approximate soln using Prandtl's method

1. For Circular Section

We have the general eqn of a circle are $x^2 + y^2 = R^2$.

The Prandtl's stress function is defined as-

$\phi = K(x^2 + y^2 - R^2)$

We have the poissons equation.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta$$

$$2K + 2K = -2G\theta$$

$$4K = -2G\theta$$

$$K = \frac{-G\theta}{2}$$

$$\phi = -\frac{G\theta}{2}(x^2 + y^2 - R^2).$$

Now we have the torque equation

$$T = 2\iint \phi\, dx\,dy.$$

$$= 2\iint -\frac{G\theta}{2}(x^2 + y^2 - R^2)\, dx\,dy$$

Variation of 'x' is from $-\sqrt{R^2-y^2}$ to $\sqrt{R^2-y^2}$

Variation of 'y' is from $-R$ to R.

$$T = -G\theta \int_{-R}^{R} \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} (x^2 + y^2 - R^2)\, dx\,dy.$$

$$= -G\theta \int_{-R}^{R} \left[\left[\frac{x^3}{3} + y^2 x - R^2 x \right]_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \right.$$

$$= G\theta \int_{-R}^{R} \left[(R^2 - y^2)x - \frac{x^3}{3} \right]_{(R^2-y^2)^{1/2}}^{(R^2+y^2)^{1/2}}$$

$$= \cancel{G\theta \int (R^2 - y^2)}$$

$$= G\theta \int_{-R}^{R} \left[(R^2-y^2) - \left(\frac{R^2-y^2}{3} \right)^{3/2} \right] - \left[-(R^2-y^2)^{3/2} + (R^2-y^2) \right]$$

$$= \frac{4}{3} G\theta \int_{-R}^{R} (R^2-y^2)^{3/2} \, dy.$$

$$T = G\theta \left(\frac{4}{3} \right) \int_{-R}^{R} (R^2-y^2)^{3/2} \, dy$$

$$= G\theta \times \frac{4}{3} \times \frac{1}{4} \left[y(R^2-y^2)^{3/2} + 3/2 R^2 y \sqrt{R^2-y^2} + 3/2 R^4 \sin^{-1}(y/R) \right]_{-R}^{R}$$

$$= G\theta \times \frac{4}{3} \times \frac{1}{4} \left[0 + 0 + 3/2 R^4 \pi \right]$$

$$= \frac{G\theta \pi R^4}{2}$$

$$\boxed{T = G\theta \frac{\pi R^4}{2}}$$

Area of the element $= dx\,dy$
$= 2\pi r . dr$

Torque, $T = 2 \iint \phi \, dx\,dy$

$$= 2 \int_A -\frac{G\theta}{2} (x^2+y^2-R^2) \, dx\,dy$$

$$= + G\theta \int_0^R (R^2 - r^2) \, 2\pi r\,dr$$

$$= 2\pi G\theta \left[\frac{R^2 r^2}{2} - \frac{r^4}{4} \right]_0^R$$

$$= 2\pi G\theta \left[\frac{R^4}{2} - \frac{R^4}{4} \right]$$

$$= \cancel{2}\pi G\theta \left[\frac{R^4}{\cancel{4}2} \right]$$

$$= G\theta \pi \left[\frac{R^4}{2} \right]$$

where $R = D/2$.

$$T = G\theta \pi \left[\frac{D^4}{32} \right]$$

$$= G\theta \frac{\pi D^4}{32}$$

$$\boxed{T = G\theta J}$$

★ Relation b/w the gradiants of ψ & ϕ

From prandtl's method we have

$$\left. \begin{aligned} \tau_{xz} &= \frac{\partial \phi}{\partial y} \\ \tau_{yz} &= -\frac{\partial \phi}{\partial x} \end{aligned} \right\} \text{Prantl's method}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{2K}{a^2}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{2K}{b^2}$$

$$\frac{2K}{a^2} + \frac{2K}{b^2} = -2G\theta$$

$$\frac{a^2+b^2}{a^2b^2} K = -G\theta$$

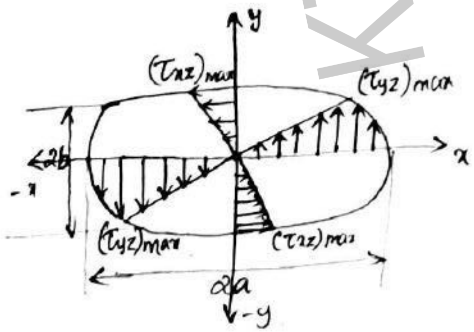
$$K = -G\theta \left[\frac{a^2b^2}{a^2+b^2}\right]$$

$$\phi = -G\theta \frac{a^2b^2}{a^2+b^2} \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right]$$

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = -G\theta \frac{a^2b^2}{a^2+b^2} \times \frac{2y}{b^2}$$

$$= -2G\theta \frac{a^2 y}{a^2+b^2}$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = 2G\theta \frac{b^2 x}{a^2+b^2}$$



τ_{xz} is max at $y = \pm b$

τ_{yz} is max at $x = \pm a$

$$\tau_{xz}/_{max} = \frac{2G\theta a^2 b}{a^2+b^2}$$

$$\tau_{yz}/_{max} = \frac{2G\theta b^2 a}{a^2+b^2}$$

By comparing equ it is clear that among the true value of τ_{xz} is max. Since $a > b$

Resultant stress $\boxed{\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}}$

J integral is

$$J = \frac{-1}{G\theta}\iint\left(x \cdot \frac{\partial\phi}{\partial x} + y \cdot \frac{\partial\phi}{\partial y}\right) dx\,dy$$

$$= \frac{-1}{G\theta}\iint\left(x \cdot \frac{2kx}{a^2} + y \cdot \frac{2ky}{b^2}\right) dx\,dy$$

$$= \frac{-1}{G\theta}\left[\frac{2k}{a^2}\iint x^2\,dx\,dy + \frac{2k}{b^2}\iint y^2\,dx\,dy\right]$$

$$= \frac{-1}{G\theta}\left[\frac{2k}{a^2} I_{yy} + \frac{2k}{b^2} \cdot I_{xx}\right]$$

$$J = \frac{-1}{G\theta}\left[\frac{-G\theta a^2 b^2}{a^2+b^2}\right] \times \frac{\pi ab}{2}$$

$$= \frac{\pi a^3 b^3}{a^2+b^2}$$

Torsion, $T = G\theta J$

$$= G\theta \frac{\pi a^3 b^3}{a^2+b^2}$$

* An elliptical shaft of semi axis $a = .05\,m$, $b = .025$ & $G = 80\,GPa$ is subjected to a twisting moment of 1200π Nm. Determine the maximum shear stress and angle of twist per unit length.

$a = .05\,m$ $\quad G = 80\,GPa = 80 \times 10^3$ N/mm²

$b = .025\,m$ $\quad T = 1200\pi$

$$\tau_{xz} = 2G\theta \frac{a^2 b}{a^2+b^2}$$

$$= 2 \times 80 \times 10^3 \times 2.4 \times 10^{-5} \times \frac{50^2 \times 25}{(50)^2 + (25)^2}$$

$$= 76.8\ N/mm^2$$

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$$

$$\theta = \frac{T}{GJ}$$

$$= \frac{1200\pi \ldots}{80 \times 10^3 \times \pi \ldots}$$

$$= 2.4 \times 10^{-5}\ rad$$

$$\tau_{yz} = \frac{2G\theta b^2 a}{a^2+b^2}$$

$$= \frac{2 \times 80 \times 10^3 \times 2.4 \times 10^{-5} \times (25)^2 \times (50)}{(50)^2 + (25)^2}$$

$$= 38.4\ N/mm^2$$

Saint Venant's Method

Shape	Warping fn ψ	$\tau_{xz} = G\theta\left(\frac{\partial\psi}{\partial x} - y\right)$	$\tau_{yz} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right)$	Resultant $\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$	Max. value of shear stress	$J = \iint\left(x^2+y^2+x\cdot\frac{\partial\psi}{\partial y} - y\cdot\frac{\partial\psi}{\partial x}\right)dxdy$	Stress Variation
$x^2+y^2=R^2$ 1. Circular	$\psi = c$.	$-G\theta y$	$G\theta x$.	$= G\theta\sqrt{x^2+y^2}$ $= G\theta R$.	$G\theta R$.	$J = \frac{\pi D^4}{32}$	
2. Elliptical		$= G\theta(A-1)y$. $= -\frac{2G\theta a^2 y}{b^2+a^2}$.	$= G\theta(A+1)x$ $= \frac{2G\theta b^2 x}{a^2+b^2}$.	$= \sqrt{\tau_{xz}^2+\tau_{yz}^2}$	$\frac{2G\theta a^2 b}{b^2+a^2}$	$J = \frac{\pi a^3 b^3}{a^2+b^2}$	

Prandtl's Method

Shape	Stress function, ϕ	$\tau_{xz} = \frac{\partial\phi}{\partial y}$	$\tau_{yz} = -\frac{\partial\phi}{\partial x}$	Resultant $\tau = \sqrt{\tau_{xz}^2+\tau_{yz}^2}$	Max. Value of shear stress	$J = \iint\left(x\cdot\frac{\partial\phi}{\partial x} + y\cdot\frac{\partial\phi}{\partial y}\right)dxdy$	
1. Circular.	$\phi = m(x^2+y^2-R^2)$ $= -\frac{G\theta}{2}(x^2+y^2-R^2)$	$-G\theta y$	$G\theta x$	$G\theta R$.	$G\theta R$.	$J = \frac{\pi D^4}{32}$.	
2. Elliptical.	$\phi = m\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right]$ $m = -\frac{G\theta a^2 b^2}{a^2+b^2}$.	$= -\frac{2G\theta a^2 y}{a^2+b^2}$.	$= \frac{2G\theta b^2 x}{a^2+b^2}$	$= \sqrt{\tau_{xy}^2+\tau_{yz}^2}$.	$\frac{2G\theta a^2 b}{a^2+b^2}$.	$J = \frac{\pi a^3 b^3}{a^2+b^2}$	

Membrane Analogue

Torsion problem of a homogenous bar is analogues with mathematically with several other physical problems. These includes the membrane under constant pressure.

The governing equation for torsion of a prismatic bar is terms of Prandtl's stress function is given by.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta \quad \text{(Poisson's equ)}$$

θ - angle of twist per unit length

G - modulus of rigidity.

Analytic solution of poisson's equation of torsion of prismatic bars is difficult to obtain.

An experimental techniques for determination of torsion consist of using a membrane like thin rubber sheet of shape same as that of cross section of the bar in torsion. This membrane is streched by the application of a uniform tension 'F' and fixed at its boundary. The membrane subjected to a uniform pressure 'p' normal to oneside of the membrane and the governing equ is written as

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{-P}{F}$$

where z - deflection of the membrane.

The value of F & P are adjusted in such a way that

$$\frac{P}{F} = 2G\theta$$

Now the deflection z of the membrane is same as that of the prandtl's stress function of the given torsion problem.

Thin Walled Section/ Tubes

A Member is said to be thin walled only if its thickness is less than $\frac{1}{15}$ of its internal diameter. Otherwise the member is said to be thick walled. In the case of thin walled section, the weight is comparitively small and hence it can be used in applications where weight reduction got has on higher priority (Aerospace industry)

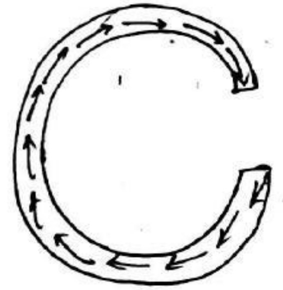
Thin walled sections are classified into two based on the shear flow circuit.

1. Thin closed wall section
2. Thin open wall section.

Thin closed wall section

A Thin walled member is said to be closed only if the complete is atleast one shearflow circuit.

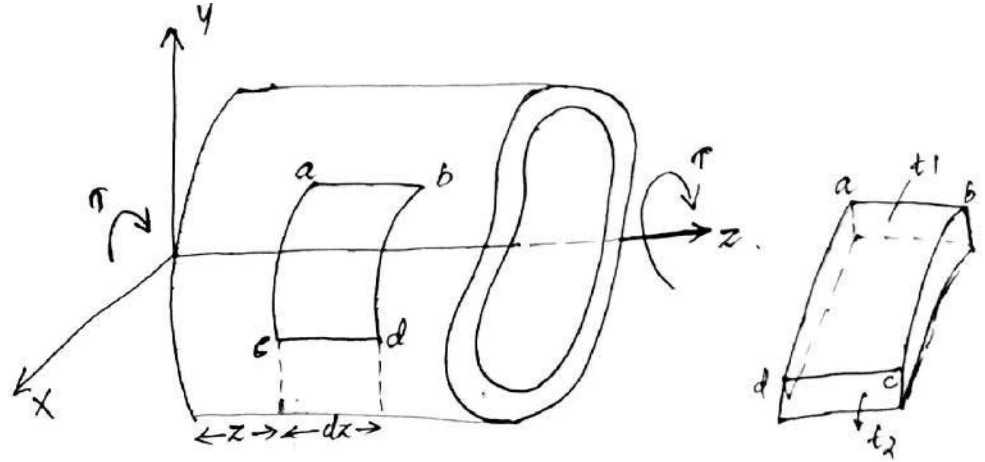
Thin open wall membrane ; This system can't complete a single shear flow - circuit



Compare two sections one closed & other open is the same material & same area of cross section. closed section resist much higher - torque than the open section. ~~is~~ resistance - to torsion is larger in both section. Deformation reduced in the closed wall section is less compared to open wall section

Torsion of thin walled section

Consider a thin closed wall section of varying thickness as shown in figure. It assumped two be of same cross section.



Now we will take a elt strip abcd of length 'dz' as shown in figure.

Let 'T_1' be the thickness on phase AB & 'T_2' be the thickness CD.

When we apply a torque 'T' developes a shear stress in cross section of the meterial. which may defer depending upon the thickness. Let 'τ_1' be the stresses developed in a section at thickness 't_1' & 'τ_2' be the corresponding stress developed at section 't_2'.

The force developed on phase AB = force devolped on phase CD

$$\tau_2 A_2 = \tau_1 A_1$$

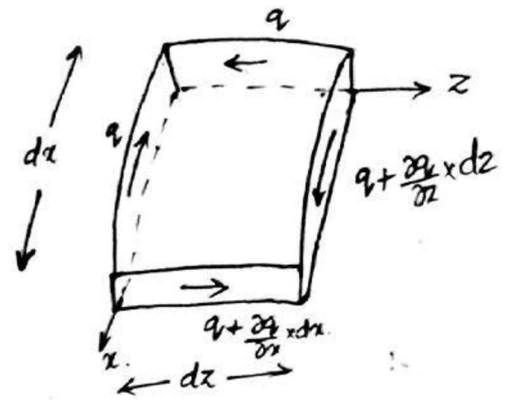
$$(\tau_2)\, t_2\, dz = (\tau_1) \times t_1 \times dz$$

$$\tau_2 t_2 = \tau_1 t_1$$

(The pdt of thickness of the two and corresponding shear stress produced on that thickness is constant at every point on the cross section. This ... known as shear flow & denoted by 'q'.

ii. $\boxed{q = \tau \times t}$

Now, Assume that the shear flow is varying with respect to x and z coordinate as shown in fig.



$\Sigma f_z = 0$

$$-q\,dz + \left(q + \frac{\partial q}{\partial x}\,dx\right)dz = 0$$

$$\frac{\partial q}{\partial x} = 0 \quad —(2)$$

$\Sigma f_x = 0.$

$$q\,dx - \left(q + \frac{\partial q}{\partial z}\times dz\right)dx = 0.$$

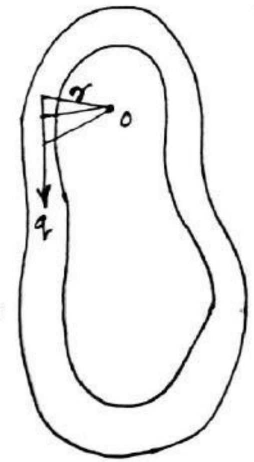
$$\left(\frac{dq}{dz}\times dz\right)dx = 0$$

$$\frac{dq}{dz} = 0 \quad —(1)$$

from equ -(1) & (2)

It is clear that the variation of shear force is zero along x & z directions

★ Expression for torque



To relate the shear flow to the applied torque 'T' consider to a infinitesimal section at a distance r from the centroid 'o', measured normal to the tangent.

Let 'q' be the shear flow developed in the section. Then force is given by.

$$dF = q\,ds$$

Now the torque on the segment,

$$dT = dF \times \perp^{ar} \text{distance}$$

$$= q\,ds \times r$$

$$= qr\,ds \;..$$

$$\therefore \quad T = \oint dT$$

$$= \oint q r \, ds$$

$$T = q \oint r \, ds$$

Mathematically, $\oint r \, ds = 2Am$

$$\therefore \quad T = q(2Am)$$

$$T = 2q\,Am$$

$$T = 2\tau t\,Am$$

$$\boxed{\tau = \frac{T}{2t \cdot Am}}$$

$Am \Rightarrow$ Area of cross section enclosed by medium length.

★ Expression for strain energy due to shear stress

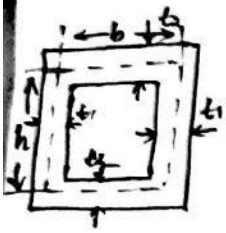
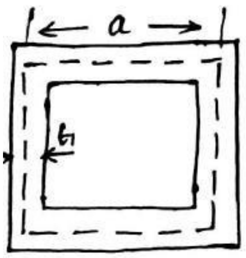
we have strain energy,

$$U = \oint \frac{\tau^2}{2G} dV$$

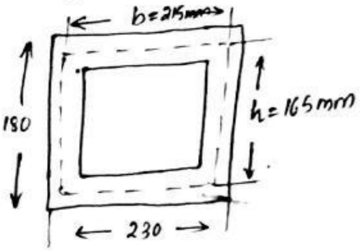
$$= \oint \frac{T^2}{4t^2 A_m^2} \cdot \frac{1}{2G} \, l t \, ds$$

$$\boxed{U = \frac{T^2 l}{8 A_m^2 G} \oint \frac{ds}{t}}$$

If $t = 0$

Shape	A_m	S	$J = \frac{4A_m^2}{\oint \frac{ds}{t}}$ or $\frac{4A_m^2 t}{S}$	$\tau = \frac{T}{2tA_m}$	$\theta = \frac{T}{GJ}$	$T = 2qA_m$
	πr^2	$2\pi r$	$J = \frac{4\pi^2 r^4 \times t}{2\pi r} = \underline{2\pi r^3 t}$	$\tau = \frac{T}{2\pi r^2 t}$	$\theta = \frac{T}{G(2\pi r^3 t)}$	$T = 2q\pi r^2$
	bh	$2(b+h)$	$J = \frac{4b^2h^2}{\frac{b}{t_2}+\frac{h}{t_1}+\frac{b}{t_2}+\frac{h}{t_1}} = \frac{4b^2h^2}{2(\frac{h}{t_1}+\frac{b}{t_2})} = \frac{4b^2h^2 t_1 t_2}{2(ht_2+bt_1)}$; $J = \frac{2b^2h^2t_1t_2}{ht_2+bt_1}$; if $t_1 = t_2 = t$, $J = \frac{2b^2h^2t}{(b+h)}$	$\tau_H = \frac{T}{2t_2 bh}$; $\tau_V = \frac{T}{2t_1 bh}$	$\theta = \frac{T(bt_1+ht_2)}{2Gb^2h^2t_1t_2}$	$T = 2qb$
	a^2	$4a$	$J = \frac{4a^4 t}{4a} = \underline{a^3 t}$	$\tau_V = \tau_H = \frac{T}{2ta^2}$	$\theta = \frac{T}{Ga^3 t}$	$T = 2qa^2$

* Estimate the shear stress ... rectangular section of size 230 mm x 180 mm, with wall thickness of 15 mm. Subjected to torque of 10 kN/m.



$b = 230 - t$
$= 230 - 15$
$= 215$ mm

$h = 180 - 15$
$= 165$ mm

$T = 10 kNm = 10 \times 10^3 \times 10^3$ Nmm
$= 10^7$ Nmm

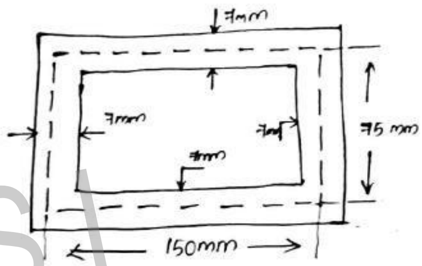
$A_m = bh$
$= 215 \times 165$
$= 35475$ mm^2

$$\tau_H = \tau_V = \frac{T}{2t A_m}$$

$$= \frac{10^7}{2 \times 215 \times 165 \times 15}$$

$= 9.396$ N/mm^2

* A hollow section shown in figure is designed for a maximum shear stress of 40 MPa. Find the twisting moment that can be taken up by the section and the angle of twist. neglect the stress concentration effect. If the section is redesigned as a hollow circular section of thickness 12 mm. find the diameter to take up the same twisting moment.



$T = 2q A_m$
$= 2\tau t A_m$

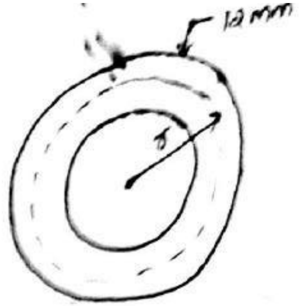
$A_m = bh$
$= 150 \times 75$
$= 11250$ mm^2

$b = 150$ mm
$h = 75$ mm

$\tau = 40$ MPa
$= 40$ N/mm^2

$t = 7$ mm

$T = 2\tau t A_m$
$= 2 \times 40 \times 7 \times 11250$
$= 63 \times 10^5$ Nmm



$T = 63 \times 10^5$ Nmm

$\tau = 40$ MPa

$t = 12$ mm

$T = 2q\,A_m$

$= 2\tau t A_m$

$63 \times 10^5 = 2 \times 40 \times 12 \times \pi r^2$

$\frac{63 \times 10^5}{2 \times 40 \times 12 \times \pi} = r^2$

$r = 45.70$ mm

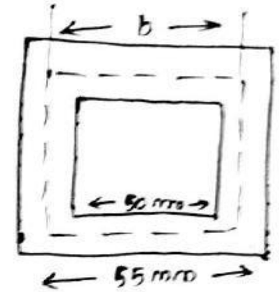
Inner diameter, $D = 2r - t$

$= 2 \times 45.7 - 12$

$= \underline{\underline{79.40 \text{ mm}}}$

★ A shaft of square section of outer side 55 mm and inner side 50 mm is subjected to a twisting moment such that the maxi. shear stress developed is 250 N/mm². What is the torque acting on the shaft and what is the angle of twist

If the shaft is 1.6 m long and G value is 70000 N/mm²



$\tau = 250$ N/mm²

$G = 70000$ N/mm²

$t = 2.5$ mm

$b = 55 - 2.5$

$= \underline{\underline{52.5 \text{ mm}}}$ (a)

$h = \underline{\underline{52.5 \text{ mm}}}$

$A_m = a^2$

$= (52.5)^2$

$= \underline{\underline{2756.25 \text{ mm}^2}}$

$S = 4a$

$= 4 \times 52.5$

$= \underline{\underline{210 \text{ mm}}}$

$T = 2q\,A_m$

$= 2\tau t A_m = 2 \times 250 \times 2.5 \times 2756.25$

$= 3445312.5$ Nmm

Angle of twist per unit length, $\theta = \dfrac{TS}{4GA_m^2t}$

Total angle of twist, $\theta_L = \dfrac{TSL}{4GA_m^2t}$

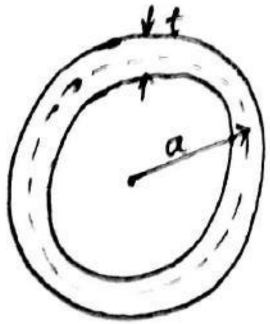
$= \dfrac{3.44\times10^6 \times 210\times1.6\times10^3}{4\times70000\times(2756.25)^2\times 2}$

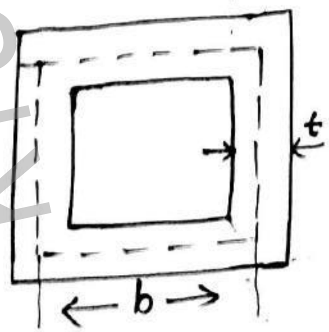
$= .2173$ radian.

★ figure shows the cross sections of two tubular holes. The thickness and circumference of the two sections are equal. find the ratio of shear stress induced. If equal

1, equal twisting moment

2, equal angle of twist are applied.





$S_c = 2\pi a$ $S_s = 4b$

$S_c = S_s$

$\cancel{2}\pi a = \cancel{4}_2 b$

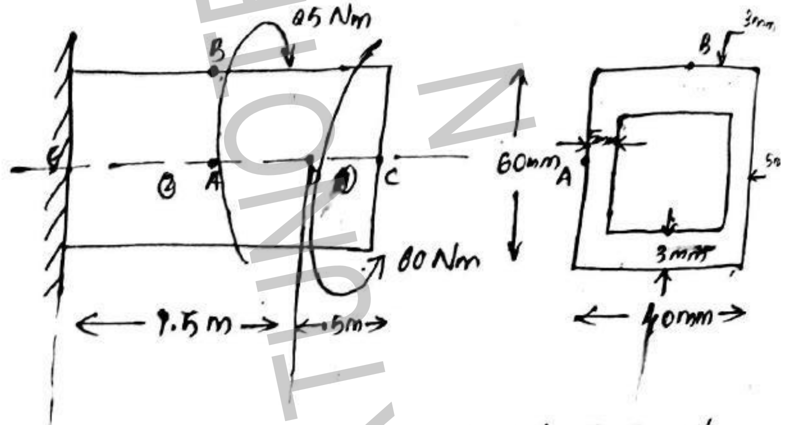
$b = \dfrac{\pi a}{2}$

$$\frac{\tau_s}{\tau_c} = \frac{A_s}{A_c}\frac{S_c}{S_s} \qquad (S_c = S_s)$$

$$= \pi/4 \times 1$$

$$= \pi/4$$

★ A Rectangular tube shown in figure is subjected to 2 torques. Determine the average shear stress at points A & B, the point 'B' is located on the top surface. What is the angle of twist at the free end, Take $G = 38 \times 10^9 \ N/m^2$



$$G = 38 \times 10^9 \ N/m^2 = 38 \times 10^9 \times 10^{-6} \ N/mm^2$$

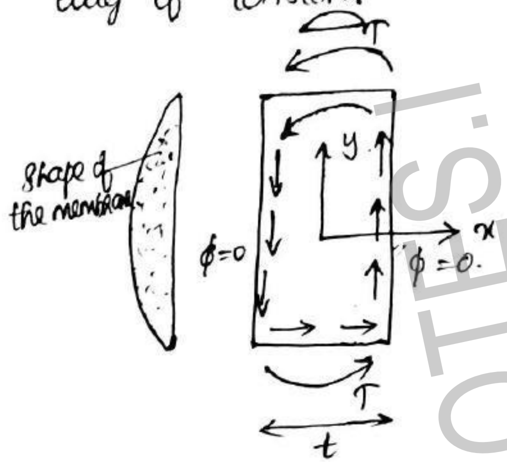
$$= 38 \times 10^3 \ N/mm^2$$

$$\tau_A = \frac{T}{2 t_A A_m}$$

$$\tau_B = \frac{T}{2 t_B A_m}$$

the width 'b' In the sense that $t \leq b/10$.

Based on membrane analogy, In thin rectangular section, Consider a membrane such as soft film which is attached to the boundary. Here Intersection of Vertical plane is assume to act through the Centre. The shear force or shear stress are flow in the same way of torsion.



The deflection of the membrane at the Centre reagion is Constant and independent of y. from membrane analogy we know that deflection is analogues to prandtl's stress function ϕ. Here deflection is independent of y and hence prandtl's stress function is also independent of y. We have the poissons equation

$$\nabla^2\phi = -2G\theta$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta$$

Now we know 'ϕ' is independent of 'y'

$$\boxed{\frac{\partial^2\phi}{\partial x^2} = -2G\theta} \quad -(1)$$

Also we have

$$\tau_{zx} = \frac{\partial \phi}{\partial y}$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x}$$

Now integrating equ – (1)

$$\frac{\partial \phi}{\partial x} = -2G\theta x + C_1$$

again integrating

$$\phi = -\cancel{2}G\theta \frac{x^2}{\cancel{2}} + C_1 x + C_2$$

$$\phi = -G\theta x^2 + C_1 x + C_2$$

Boundary conditions at $x = \pm t/2$, $\phi = 0$

$x = t/2$ $$0 = -G\theta \frac{t^2}{4} + C_1 t/2 + C_2$$

at $x = -t/2$

$$0 = -G\theta \frac{t^2}{4} - C_1 t/2 + C_2$$

$$-G\theta t^2/4 + C_1 t/2 + C_2 = 0$$

$$-G\theta t^2/4 - C_1 t/2 + C_2 = 0$$

$$0 + \cancel{2}C_1 t/\cancel{2} = 0$$

$$C_1 = 0$$

$$C_2 = G\theta \frac{t^2}{4}$$

$$\therefore \phi = -G\theta x^2 + G\theta \frac{t^2}{4}$$

Now we have torque $T = 2\iint \phi \, dx\, dy$.

$$T = 2\iint (-G\theta x^2 + G\theta t^2/4)\, dx\, dy$$

$$= 2\int_{-t/2}^{t/2} (-G\theta x^2 + G\theta t^2/4)\, dx \cdot \int_{-b/2}^{b/2} dy$$

$$= 2\left[\left[-\frac{G\theta x^3}{3} + G\theta \frac{t^2}{4} x\right]_{-t/2}^{t/2} \cdot \left[y\right]_{-b/2}^{b/2}\right]$$

$$= 2\left[\left(-\frac{G\theta t^3}{24} + \frac{G\theta t^3}{12}\right) - \left(\frac{G\theta t^3}{24} - \frac{G\theta t^3}{12}\right) \cdot \left[b/2 - -b/2\right]\right]$$

$$= 2\left[\left(\frac{\cancel{2}G\theta t^3}{\cancel{12}_6}\right) \cdot (b)\right]$$

$$= \frac{G\theta t^3}{3} b. \qquad = G\theta \frac{t^3 b}{3}$$

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = 0$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = 2G\theta x$$

$$\tau_{yz}\big|_{max} = \cancel{2}G\theta t/\cancel{2} \qquad (x = t/2)$$

$$= G\theta t$$

$$\boxed{\begin{aligned}\tau_{yz}/_{max} &= G\theta t \\ &= \frac{3T}{bt^2}\end{aligned}}$$

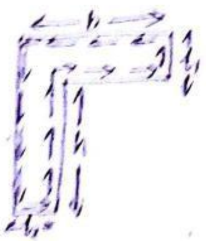
$G\theta t = \frac{3T}{bt^2}$

$\frac{3T}{bt^2}$

★ Torsion of rolled Section

The assumptions used for finding the J integral of a thin rectangular section, can be extended to a rolled section. Can be considered as union of these sections and eqn. $J = \frac{1}{3}bt^3$ each of thin rectangles can be summed up to find J value for the entire sections.

Torque transmitted by the box equal to drag

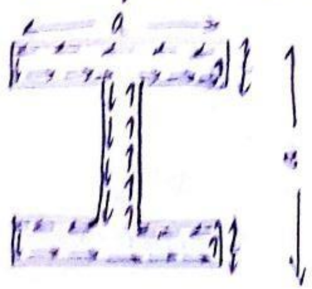


$J = J_1 + J_2$

$= \frac{1}{3}b_1 t_1^3 + \frac{1}{3} b_2 t_2^3$

$J = \sum_{i=1}^{n} \frac{b_i t_i^3}{3}$

★ Find the J value of the section shown in fig.



$J_1 = \frac{1}{3} a t^3$

$J_2 = \frac{1}{3}(a - 2t) \cdot t^3$

$J_3 = \frac{1}{3} a t^3$

$J = \frac{1}{3}at^3 + \frac{1}{3}at^3 - \frac{2}{3}t^4 + \frac{1}{3}at^3$

$= at^3 - \frac{2}{3}t^4$

★ Multiply Connected Sections

A cross section is said to be multiply connected. If more than one closed curves in its cross section, ie a box with more than one holes in the cross section is multiply connected sections.

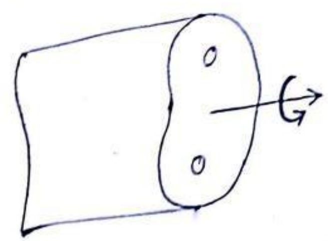


Fig : shows a multiply connected section with one exterior boundary C_1 and 2 interior boundaries C_2, C_3.

We have $T = \iint (x \cdot \tau_{yz} - y\, \tau_{xz})\, dx\, dy$

$= \iint \left(-x \cdot \frac{\partial \phi}{\partial x} - y \cdot \frac{\partial \phi}{\partial y}\right) dx\, dy$

$= \int x \phi \, dy - \ldots$

(Major expression for torque in terms of Prandtl's).

Here ϕ is not constant over the cross section.

$$= \oint \phi (x\,dy - y\,dx) + 2 \iint \phi \, dx\,dy$$

$$= \oint \phi (y\,dx - x\,dy) - 2 \iint \phi \, dx\,dy$$

First integral of this equation is the cyclic integral over the boundary curves of the section.

In multiply connected sections,

$\oint \phi (y\,dx - x\,dy)$ is the sum of circular integral over the boundaries.

★ Green's theorem

$$\oint (y\,dx - x\,dy) = 2A_i$$

where A – Area bounded by the curves

i – number of boundaries

∴ Torque equ can be rewritten as

$$T = 2 A_i \phi_i + 2 \iint \phi \, dx\,dy$$

★ Boundary condition of 'ϕ' for a bar multiple boundaries.

The value for 'ϕ' for the outer boundary curves can be fixed as zero (By membrane analogy) inner boundary is smooth & hence its curvature

$\frac{\partial \phi}{\partial n}$ is constant

Now we have

$$\oint \frac{\partial \phi}{\partial n} \cdot ds = 2G\theta A_i$$

$$\frac{\partial \phi}{\partial n} \oint ds = 2G\theta \times \pi R^2$$

$$\frac{\partial \phi}{\partial n} \times 2\pi R = 2G\theta \pi R^2$$

$$\frac{\partial \phi}{\partial n} = G\theta R$$

$$\frac{\partial \phi}{\partial n} = \frac{\phi_1 - \phi_2}{t} = G\theta R$$

$$\frac{\phi_1}{t} = G\theta R$$

$$\phi_1 = G\theta R t$$

We have $T = 2A_i\phi_1 + 2\iint \phi \, dx dy$

Since thickness is very small compared to the radius, the 2nd term on the above eqn can be neglected

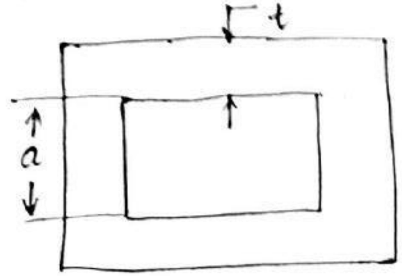
$$T = 2A_i\phi_1 + \cancel{2A\phi_2}$$

$$= 2\pi R^2 \times G\theta R t$$

$$= G\theta \times 2\pi R^3 t$$

$$= G\theta J \qquad J = 2\pi R^3 t$$

★ Find the J-integral of the section shown in fig. Assume that stress function varies linearly across the section



Assume

ϕ_1 – Prandtl's stress fn of inner surface

ϕ_2 – Prandtl's stress fn of outer surface

outer surface is fixed,

$\therefore \phi_2 = 0$

inner surface is smooth

$\therefore \frac{\partial \phi}{\partial n} = $ a const.

$$\oint \frac{\partial \phi}{\partial n} ds = 2G\theta A_i$$

$$\frac{\partial \phi}{\partial n} \oint ds = 2G\theta \times a^2$$

$$\frac{\partial \phi}{\partial n} \times 4a = 2G\theta \times a^2$$

$$\frac{\partial \phi}{\partial n} = G\theta \frac{a}{2}$$

$$\frac{\phi_1 - \phi_2}{t} = G\theta\, a/2$$

$$\phi_1 = G\theta \frac{at}{2}$$

$$T = 2A_i\phi_i + 2\iint \phi \, dx\,dy$$

$$= 2A_1\phi_1 + 2A_2\phi_2$$

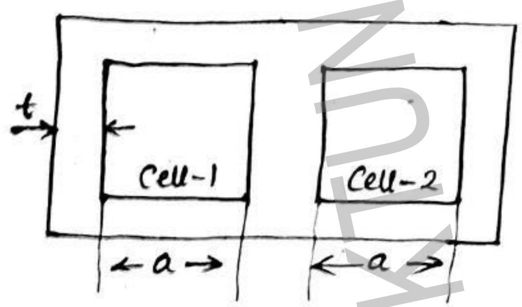
$$= 2a^2 \times G\theta \frac{a}{2} t$$

$$= G\theta a^3 t$$

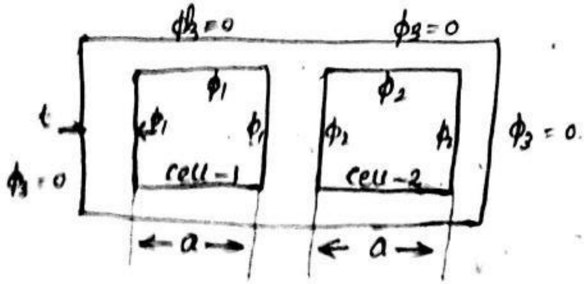
$$= G\theta J$$

$$J = a^3 t$$

★ Find the torsional shear stress and J-integral of the section shown in fig. Assume $t \ll a$



Ans



Cell – 1

$$\oint \frac{\partial \phi}{\partial n}\, ds = 2G\theta A_i$$

$$\left[\frac{\phi_1 - \phi_3}{t} \times 3a\right] + \left(\frac{\phi_1 - \phi_2}{t} \times a\right) = 2G\theta a^2$$

$$\frac{\phi_1}{t} \times 3a + \frac{\phi_1 - \phi_2}{t} a = 2G\theta a^2$$

$$\cancel{a}\left(\frac{\phi_1}{t} \times 3 + \frac{\phi_1 - \phi_2}{t}\right) = 2G\theta a^{\cancel{2}}$$

$$\frac{\phi_1}{t} \times 3 + \frac{\phi_1 - \phi_2}{t} = 2G\theta a$$

$$4\phi_1 - \phi_2 = 2G\theta at \quad \text{—(1)}$$

Case – 2

$$\oint \frac{\partial \phi}{\partial n}\, ds = 2G\theta A_i$$

$$\left(\frac{\phi_2 - \phi_3}{t} \times 3\cancel{a} + \frac{\phi_2 - \phi_1}{t} \times \cancel{a}\right) = 2G\theta a^{\cancel{2}}$$

$$\left(\frac{3\phi_2}{t} + \frac{\phi_2 - \phi_1}{t}\right) = 2G\theta a$$

$$4\phi_2 - \phi_1 = 2G\theta at \quad \text{—(2)}$$

equ – (1) × 4

$$16\phi_1 - 4\phi_2 = 8G\theta at \quad +$$

$$-\phi_1 + 4\phi_2 = 2G\theta at$$

$$15\phi_1 = 10G\theta at$$

$$\phi_1 = \frac{10}{15} G\theta a t$$

$$= \frac{2}{3} G\theta a t$$

$$\phi_2 = \frac{2}{3} G\theta a t$$

$$T = 2A_1\phi_1 + 2A_2\phi_2$$

$$= 4A_1\phi_1$$

$$= 4a^2 \times \frac{2}{3} G\theta a t$$

$$= \frac{8}{3} G\theta a^3 t \qquad J = \frac{8}{3} a^2 t$$

Shear stress of 'ϕ' left and bottom edges of the cell-I is given by

$$\tau = \frac{\partial \phi}{\partial n} = \frac{\phi_1 - \phi_3}{t_1} = \phi_1/t$$

$$= \frac{2}{3} G\theta a$$

Shear stress on the right edges of the cell-I = shear stress on left stress of cell-2 which is equal to shear stress of central edge.

$$t = \frac{\partial \phi}{\partial n} = \frac{\phi_1 - \phi_2}{4} = 0$$

Shear stress on right & bottom of cell-II

$$\tau = \frac{\partial \phi}{\partial n} = \frac{\phi_2 - \phi_3}{t} = \frac{\phi_2}{t}$$

$$= \frac{2}{3} G\theta a$$

$$= \frac{\pi a b}{4} \times \frac{\ldots}{b^2+a^2}$$

$$= \frac{\pi a^3 b^3}{b^2+a^2}$$

$$= \iint [x^2+y^2+Ax^2-Ay^2]\, dx\, dy$$

$$= (1+A)\iint x^2\, dx\, dy + (1-A)\iint y^2\, dx\, dy$$

$$= (1+A) I_{yy} + (1-A) I_{xx}$$